Name________________________

Student I.D.__________________

Math 2250-4
Quiz 10 SOLUTIONS
April 6, 2012

Directions: Because the homework was split evenly between Laplace transforms and eigenvectors this week, <u>you may choose either problem 1 or problem 2 below to complete</u>. If you attempt both, make it very clear which one you want graded.

1) Consider the matrix

$$A := \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}.$$

a) Find the eigenvalues and eigenvectors (eigenspace bases).

(8 points)

b) Is this matrix diagonalizable? Explain why or why not.

(2 points)

<u>a)</u>

$$\left| A - \lambda I \right| = \begin{vmatrix} 3-\lambda & 4 \\ -1 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) + 4 = (\lambda-3)(\lambda+2) + 4 = \lambda^2 - \lambda - 6 + 4$$
$$= \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$$

So the eigenvalues are $\lambda = 2, -1$.

<u>Eigenvectors:</u>

$\lambda = 2$: *The homogeneous system* $(A - 2I)\underline{\mathbf{v}} = \underline{\mathbf{0}}$ *has augmented matrix*

$$\left[\begin{array}{cc|c} 1 & 4 & 0 \\ -1 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \underline{\mathbf{v}} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} \text{ is a basis for } E_2 .$$

$\lambda = -1$: *The homogeneous system* $(A + I)\underline{\mathbf{v}} = \underline{\mathbf{0}}$ *has augmented matrix*

$$\left[\begin{array}{cc|c} 4 & 4 & 0 \\ -1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \underline{\mathbf{v}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ is a basis for } E_{-1}.$$

<u>b)</u> *The matrix <u>is</u> diagonalizable because the two eigenvectors above are a basis for* $\mathbb{R}^2$ *(This suffices for an answer). As a consequence (and this is why the condition is called diagonalizable), the identity*

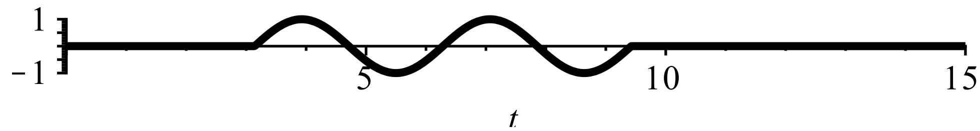
$$AP = PD \Rightarrow A = PDP^{-1}$$

holds, where P is the matrix that has the two eigenvectors in its columns, and D is the corresponding diagonal matrix of eigenvalues.

2) Solve the initial value problem below for an undamped mass-spring configuration subject to impulse forces at time $t = \pi$ and $t = 3\pi$. (There is a Laplace transform table on the back of this quiz.)

$$x''(t) + 4x(t) = 2\,\delta(t-\pi) - 2\,\delta(t-3\pi)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

Hint: The solution has this graph:



(10 points)

Solution:

Taking the Laplace transform of the IVP yields

$$s^2X(s) + 4X(s) = 2e^{-\pi s} - 2e^{-3\pi s}$$
$$\Rightarrow X(s)\left(s^2+4\right) = 2e^{-\pi s} - 2e^{-3\pi s}$$
$$\Rightarrow X(s) = e^{-\pi s}\frac{2}{s^2+4} - e^{-3\pi s}\frac{2}{s^2+4}\,.$$
$$\Rightarrow x(t) = u(t-\pi)\sin(2(t-\pi)) - u(t-3\pi)\sin(2(t-3\pi))\,.$$

Since $\sin(2t)$ repeats with a period of (any integer multiple of) π, we can simplify the expression above:

$$x(t) = \left(u(t-\pi) - u(t-3\pi)\right)\sin(2t) = \begin{cases} \sin(2t), & t \in [\pi, 3\pi) \\ 0 & , t \notin [\pi, 3\pi) \end{cases},$$

And this piecewise formula for $x(t)$ agrees with the displayed graph.

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\dfrac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\dfrac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\dfrac{1}{2k^3}(\sin kt - kt\cos kt)$	$\dfrac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\dfrac{t}{2k}\sin kt$	$\dfrac{s}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\dfrac{1}{2k}(\sin kt + kt\cos kt)$	$\dfrac{s^2}{(s^2+k^2)^2}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t)$, period p	$\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\dfrac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\dfrac{1}{s}\tanh\dfrac{as}{2}$
t	$\dfrac{1}{s^2}$	$\left[\!\!\left[\dfrac{t}{a}\right]\!\!\right]$ (staircase)	$\dfrac{e^{-as}}{s(1-e^{-as})}$
t^n	$\dfrac{n!}{s^{n+1}}$		
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$		
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$		