Math 2250-4
Wed Mar 7 Note: in homework problem w9.3d take c=10 (not c=5).

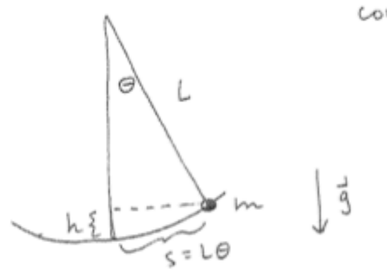
5.4 Applications of 2nd order linear homogeneous DE's with constant coefficients, to unforced spring (and related) configurations.

$$m\,x'' + c\,x' + k\,x = 0$$

- On Tuesday we discussed undamped oscillations, $c = 0$, which are described by simple harmonic motion. Today discuss damping $c \neq 0$ in the unforced oscillator equation above, following the last two pages of Tuesday's notes. We wish to understand the overdamped, critically damped, and underdamped situations.

- Then we should have time for two (undamped) experiments, using a pendulum configuration in one, and an oscillating mass-spring configuration in the other.

1) pendulum



$\theta = \theta(t)$

conservative system KE + PE = const.

$\frac{1}{2}mv^2 + mgh = \text{const}$

$s = L\theta$

$v = \frac{ds}{dt} = L\theta'(t)$ $\quad h = L - L\cos\theta = L(1-\cos\theta)$

so, $\frac{1}{2}m\,L^2(\theta'(t))^2 + mgL\,(1-\cos(\theta(t))) \equiv \text{const}$

D_t: $mL^2\,\theta'\theta'' + mgL(\sin\theta)\,\theta' \equiv 0$

$\underbrace{mL\theta'}(L\theta'' + g\sin\theta) \equiv 0$

$\neq 0$ except at isolated times

$\rightsquigarrow$ deduce eqtn of motion is $\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$

linearize: $\boxed{\theta'' + \frac{g}{L}\theta = 0}$

$\omega_0 = \sqrt{\frac{g}{L}}$

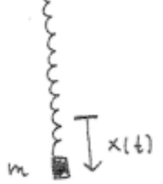
$\theta(t) = C\cos(\omega_0 t - \alpha)$

non-linear DE

but $\sin\theta = \theta - \frac{\theta^3}{3!} + \ldots$

$\sin\theta \approx \theta$ $\quad\theta$ small is excellent approx (alternating series test)

2) hanging mass-spring:



$mx'' = -kx$

$mx'' + kx = 0$

$\boxed{x'' + \frac{k}{m}x = 0}$

$\omega_0 = \sqrt{\frac{k}{m}}$

Why don't you see gravity g in this DE?

Pendulum: measurements and prediction (we'll check these numbers).

```
> restart:
  Digits := 4:
```

```
> L := 1.526;
  g := 9.806;
  ω := sqrt(g/L); # radians per second
  f := evalf(ω/(2·Pi)); # cycles per second
  T := 1/f; # seconds per cycle
```

Experiment:

Mass-spring:
compute Hooke's constant:

```
> 98.7 − 83.4; #displacement from extra 50g
```

```
> k := .05·9.806/.153; # solve k·x=m·g for k.
```

```
> m := .1; # mass for experiment is 100g
  ω := sqrt(k/m); # predicted angular frequency
  f := evalf(ω/(2·Pi)); # predicted frequency
  T := 1/f; # predicted period
```

Experiment:

We neglected the KE_{spring}, which is small but could be adding intertia to the system and slowing down the oscillations. We can account for this:

<u>Improved mass-spring model</u>
Normalize $TE = KE + PE = 0$ for mass hanging in equilibrium position, at rest. Then for system in motion,

$$KE + PE = KE_{mass} + KE_{spring} + PE_{work}.$$

$$PE_{work} = \int_0^x k\,s\,ds = \frac{1}{2}k\,x^2, \quad KE_{mass} = \frac{1}{2}m\,(x'(t))^2, \quad KE_{spring} = ????$$

How to model KE_{spring} ? Spring is at rest at top (where it's attached to bar), moving with velocity $x'(t)$ at bottom (where it's attached to mass). Assume it's moving with velocity $\mu\, x'(t)$ at location which is fraction μ of the way from the top to the mass. Then we can compute KE_{spring} as an integral with respect to μ , as the fraction varies $0 \le \mu \le 1$:

$$KE_{spring} = \int_0^1 \frac{1}{2}\left(\mu\, x'(t)\right)^2 \left(m_{spring}\, \mathrm{d}\mu\right)$$

$$= \frac{1}{2} m_{spring} \left(x'(t)\right)^2 \int_0^1 \mu^2\, d\,\mu = \frac{1}{6} m_{spring} \left(x'(t)\right)^2 \ .$$

Thus

$$TE = \frac{1}{2}\left(m + \frac{1}{3} m_{spring}\right)\left(x'(t)\right)^2 + \frac{1}{2} k x^2 = \frac{1}{2} M \left(x'(t)\right)^2 + \frac{1}{2} k x^2 ,$$

where

$$M = m + \frac{1}{3} m_{spring}$$

$\mathrm{D}_t(TE) = 0 \Rightarrow$

$$M\, x'(t)\, x''(t) + k\, x(t)\, x'(t) = 0 \ .$$
$$x'(t)\left(M\, x'' + k\, x\right) = 0 \ .$$

Since $x'(t) = 0$ only at isolated $t-$values, we deduce that the corrected equation of motion is

$$\left(M\, x'' + k\, x\right) = 0$$

with

$$\omega_0 = \sqrt{\frac{k}{M}} = \sqrt{\frac{k}{m + \frac{1}{3} m_{spring}}} \ .$$

Does this lead to a better comparison between model and experiment?

```
> ms := .011;  # spring has mass 11g
  M := m + 1/3·ms;  #  "effective mass"

> ω := sqrt(k/M);  # predicted angular frequency
  f := evalf(ω/(2·Pi));  # predicted frequency
  T := 1/f;  # predicted period

>
```