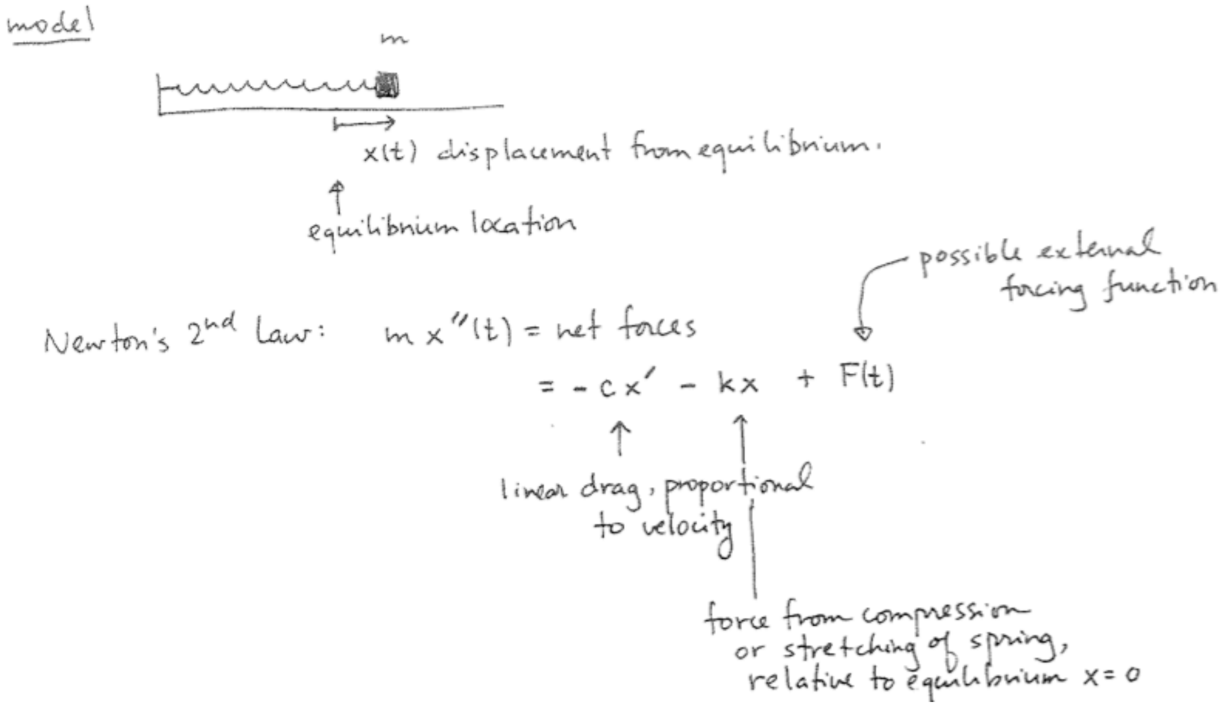
Math 2250-4
Tues Mar 6

5.4 Applications of 2^{nd} order linear homogeneous DE's with constant coefficients, to unforced spring (and related) configurations.

In this section we study the differential equation below for functions $x(t)$:

$$m\,x'' + c\,x' + k\,x = 0\,.$$



In section 5.4 we assume the external forcing function $F(t) \equiv 0$. The expression for internal forces $-c\,x' - k\,x$ is a linearization model, about the constant solution $x = 0$, $x' = 0$, for which the net forces must be zero. Notice that $c \geq 0$, $k > 0$. The actual internal forces are probably not exactly linear, but this model is usually effective when x, x' are sufficiently small. k is called the Hooke's constant, and c is called the damping coefficient.

This is a constant coefficient linear homogeneous DE, so we try $x(t) = e^{r\,t}$ and compute

$$L(x) := m\,x'' + c\,x' + k\,x = e^{r\,t}\left(m\,r^2 + c\,r + k\right) = e^{r\,t}\,p(r).$$

The different behaviors exhibited by solutions to this mass-spring configuration depend on what sorts of roots the characteristic polynomial $p(r)$ pocesses...

Case 1) no damping ($c = 0$).

$$m\,x'' + k\,x = 0$$

$$x'' + \frac{k}{m}\,x = 0\,.$$

$$p(r) = r^2 + \frac{k}{m}\,,$$

has roots

$$r^2 = -\frac{k}{m} \quad \text{i.e.} \quad r = \pm\, i\sqrt{\frac{k}{m}}\,.$$

So the general solution is

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right).$$

We write $\sqrt{\frac{k}{m}} := \omega_0$ and call ω_0 the <u>natural angular frequency</u>. Notice that its units are radians per time. We also replace the linear combination coefficients c_1, c_2 by A, B. So, using the alternate letters, the general solution to

$$x'' + \omega_0^{\,2}\,x = 0$$

is

$$x(t) = A\cos\left(\omega_0\,t\right) + B\sin\left(\omega_0\,t\right).$$

This motion is called <u>simple harmonic motion,</u> and can be described in terms of an <u>amplitude</u> C and a <u>phase angle</u> α (or in terms of a <u>time delay</u> δ). This is because of the addition angle trigonometry identities which let us rewrite $x(t)$ as

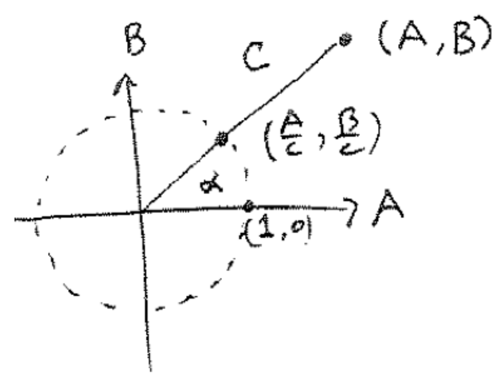
$$A\cos\left(\omega_0\,t\right) + B\sin\left(\omega_0\,t\right) = C\cos\left(\omega_0 t - \alpha\right) = C\cos\left(\omega_0\left(t - \delta\right)\right).$$

<u>Exercise 1)</u> Use the addition angle formula $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ to verify the first identity above, showing in addition that A, B are related to C, α, δ by
$A = C\cos\alpha$, $B = C\sin\alpha$, so that (working backwards):

$$C = \sqrt{A^2 + B^2}$$

$$\frac{A}{C} = \cos(\alpha),\ \frac{B}{C} = \sin(\alpha),\ \frac{B}{A} = \tan(\alpha)\,,$$

$$\delta = \frac{\alpha}{\omega_0}\,.$$

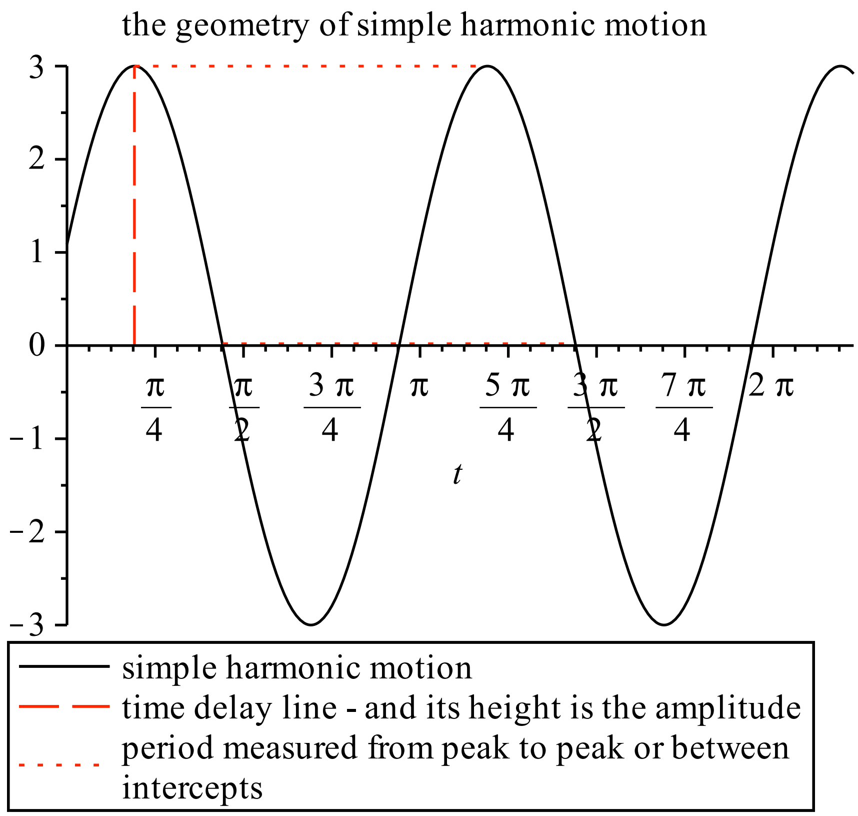


$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha) = C\cos(\omega_0 (t - \delta))$$

The amplitude C is the maximum absolute value of $x(t)$, and the time delay δ is how much the graph of $C\cos(\omega_0 t)$ is shifted to the right in order to obtain the graph of $x(t)$. Other important data is

$$f = \text{frequency} = \frac{\omega_0}{2\pi} \quad \text{cycles/time}$$

$$T = \text{period} = \frac{2\pi}{\omega_0} = \text{time/cycle.}$$



(I made that plot above with these commands...and then added a title and a legend, from the plot options.)

```
> with(plots) :
> plot1 := plot(3·cos(2(t − .6)), t = 0 ..7, color = black) :
  plot2 := plot([.6, t, t = 0 ..3.], linestyle = dash) :
  plot3 := plot(3, t = .6 ..(.6) + Pi, linestyle = dot) :
  plot4 := plot(0.02, t = .6 + Pi/4 ...6 + 5·Pi/4, linestyle = dot) :
> display({plot1, plot2, plot3, plot4});
>
```

Exercise 2) A mass of 2 *kg* oscillates without damping on a spring with Hooke's constant $k = 18\ \frac{N}{m}$. It is initially stretched 1 *m* from equilibrium, and released with a velocity of $\frac{3}{2}\ \frac{m}{s}$.

2a) Show that the mass' motion is described by $x(t)$ solving the initial value problem

$$x'' + 9\,x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

2b) Solve the IVP in a, and convert $x(t)$ into amplitude-phase and amplitude-time delay form. Check your work with the commands below.

```
> with(DEtools):
> with(plots):
> deqtn1 := x''(t) + 9·x(t) = 0:
  ICS := x(0) = 1, x'(0) = 3/2:
  dsolve({deqtn1, ICS}, x(t));
> C := sqrt(1 + .5^2);
  alpha := arctan(.5);
  delta := alpha/3;
> plot1 := plot(1/2·sin(3·t) + cos(3·t), t = 0..4, color = red):
  plot2 := plot(C·cos(3·t − alpha), t = 0..4, color = black):
  plot3 := plot(C·cos(3·(t − delta)), t = 0..4, color = blue):
  display({plot1, plot2, plot3}, title = `overlay`);
>
```

Case 2: damping

$$m x'' + c x' + k x = 0$$

$$x'' + \frac{c}{m} x' + \frac{k}{m} x = 0$$

rewrite as

$$x'' + 2 p x' + \omega_0^2 x = 0.$$

$\left(p = \frac{c}{2m}, \omega_0^2 = \frac{k}{m}\right)$. The characteristic polynomial is

$$r^2 + 2 p r + \omega_0^2 = 0$$

which has roots

$$r = -\frac{2p \pm \sqrt{4p^2 - 4\omega_0^2}}{2} = -p \pm \sqrt{p^2 - \omega_0^2}.$$

2a) ($p^2 > \omega_0^2$, or $c^2 > 4 m k$). overdamped. In this case we have two negative real roots

$$r_1 < r_2 < 0$$

and

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = e^{r_1 t}\left(c_1 + c_2 e^{(r_2 - r_1) t}\right).$$

• solution converges to zero exponentially fast; solution passes through equilibrium location $x = 0$ at most once.

2b) ($p^2 = \omega_0^2$, or $c^2 = 4 m k$) critically damped. Double real root $r_1 = r_2 = -p = -\frac{c}{2m}$.

$$x(t) = e^{-p t}\left(c_1 + c_2 t\right).$$

• solution converges to zero exponentially fast, passing through $x = 0$ at most once, just like in the overdamped case. The critically damped case is the transition between overdamped and underdamped:

2c) ($p^2 < \omega_0^2$, or $c^2 < 4 m k$) underdamped. Complex roots

$$r = -p \pm \sqrt{p^2 - \omega_0^2} = -p \pm i\,\omega_1$$

with $\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$.

$$x(t) = e^{-p t}\left(A \cos(\omega_1 t) + B \sin(\omega_1 t)\right) = e^{-p t} C \cos(\omega_1 t - \alpha_1).$$

• solution decays exponentially to zero, but oscillates infinitely often, with exponentially decaying pseudo-amplitude $e^{-p t} C$ and pseudo-angular frequency ω_1, and pseudo-phase angle α_1.

Exercise 3) Solve for $x(t)$, and classify by type of damping:

3a)

$$x'' + 6x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

> $dsolve\left(\left\{x''(t) + 6\cdot x'(t) + 9\cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right);$

>

3b)

$$x'' + 10x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

> $dsolve\left(\left\{x''(t) + 10\cdot x'(t) + 9\cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right);$

>

3c)

$$x'' + 2x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

> $dsolve\left(\left\{x''(t) + 2\cdot x'(t) + 9\cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right);$

>

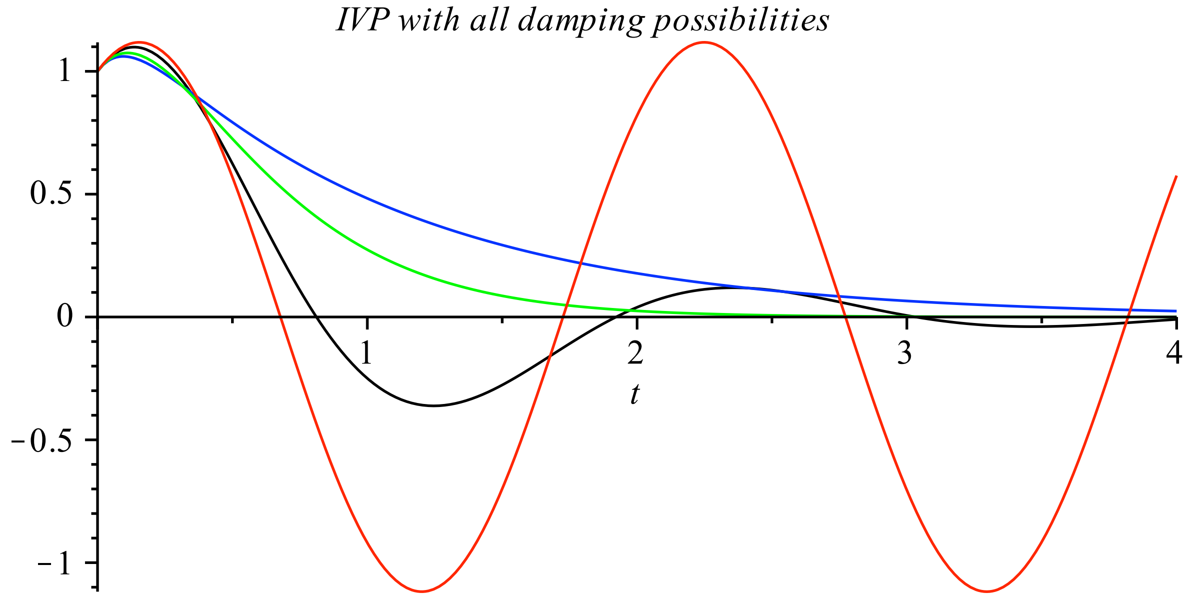
```
> plot3a := plot(exp(-3·t)·(1 + 9/2·t), t = 0..4, color = green) :
  plot3b := plot(21/16·exp(-t) - 5/16·exp(-9·t), t = 0..4, color = blue) :
  plot3c := plot(5/8·sqrt(2) e^(-t)·sin(2 sqrt(2)·t) + e^(-t)·cos(2 sqrt(2)·t), t = 0..4, color = black) :
  display({plot1, plot3a, plot3b, plot3c}, title = `IVP with all damping possibilities`);
```

$$plot3a := plot\left(\exp(-3\cdot t)\cdot\left(1+\frac{9}{2}\cdot t\right), t=0..4, color=green\right):$$

$$plot3b := plot\left(\frac{21}{16}\cdot\exp(-t)-\frac{5}{16}\cdot\exp(-9\cdot t), t=0..4, color=blue\right):$$

$$plot3c := plot\left(\frac{5}{8}\cdot\sqrt{2}\,e^{-t}\cdot\sin\left(2\sqrt{2}\cdot t\right)+e^{-t}\cdot\cos\left(2\sqrt{2}\cdot t\right), t=0..4, color=black\right):$$

display({*plot1*, *plot3a*, *plot3b*, *plot3c*}, *title* = \`*IVP with all damping possibilities*\`);



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