Math 2250-4
Mon Mar 5

5.3 How to find the solution space for n^{th} order linear homogeneous DE's with constant coefficients, and why the algorithms work.

We're focusing on the y_H part of the general solution $y = y_P + y_H$, i.e. finding the n-dimensional solution space to

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

in the case that $a_{n-1}, a_{n-2},..., a_1, a_0$ are constants.

step 1) In all cases we first try to find a basis made of exponential functions....trying $y(x) = e^{r x}$ yields

$$L(y) = e^{r x}\left(r^n + a_{n-1}r^{n-1} + ... + a_1 r + a_0\right) = e^{r x} p(r) .$$

The characteristic polynomial $p(r)$ and how it factors are the keys to finding the solution space to $L(y) = 0$. For each root r_j of $p(r)$, we get a solution $e^{r_j x}$ to the homogeneous DE.

On Friday we completed
Case 1) If $p(r)$ has n distinct (i.e. different) real roots $r_1, r_2, ..., r_n$, then

$$e^{r_1 x}, e^{r_2 x}, ... , e^{r_n x}$$

are a basis for the solution space; i.e. the general solution is given by

$$y_H(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + ... + c_n e^{r_n x} .$$

We began
Case 2) Repeated real roots. In this case $p(r)$ has all real roots $r_1, r_2, ... r_m$ $(m < n)$ with the r_j all different, but with some of the factors $(r - r_j)$ in $p(r)$ appearing with powers bigger than 1 . In other words, $p(r)$ factors as

$$p(r) = (r - r_1)^{k_1}(r - r_2)^{k_2}...(r - r_m)^{k_m}$$

with some of the $k_j > 1$, and $k_1 + k_2 + ... + k_m = n$.

We checked the smallest possible example of a second order DE and a double root. Using our usual "plug in and check" method in the second order case

$$L(y) := y'' + a_1 y' + a_0 y$$

for which $p(r)$ has a double root r_1, i.e.

$$p(r) = r^2 + a_1 r + a_0 = (r - r_1)^2 = r^2 - 2 r_1 r + r_1^2$$

it turned out by "magic" cancellation that $x e^{r_1 x}$ solves $L(y) = 0$, in addition to $e^{r_1 x}$. Any one of our favorite ways to check linear independence will quickly imply that they're a basis for the solution space to $L(y) = 0$, so the general homogeneous solution is

$$y_H(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x} .$$

There's a deeper way to understand that last computation and where it came from. This understanding helps explain the general case below and will also help us understand certain algorithms when we study non-homogeneous DE's later on, so let's talk about it: Begin with the derivative operator D defined by $D(y) := y'$. Then the second derivative operator is $D^2(y) := D \circ D(y)$, etc. So for

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y$$

we may write

$$L = D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0 I$$

where I is the identity operator, $I(y) := y$. Any factorization of the characteristic polynomial $p(r)$ leads to a factorization of L . For example, in the second order case with real roots,

$$L = D^2 + a_1 D + a_0 I$$

If $p(r)$ factors with real roots,

$$p(r) = r^2 + a_1 r + a_0 = (r - r_1)(r - r_2) = r^2 - (r_1 + r_2) r + r_1 r_2$$

Then L factors the same way:

$$L = D^2 - (r_1 + r_2) D + r_1 r_2 I = (D - r_1 I) \circ (D - r_2 I).$$

It's easy to check that

$$(D - r_1 I)\left(e^{r_1 x}\right) = 0, \; (D - r_2 I)\left(e^{r_2 x}\right) = 0$$

and that's one reason why $e^{r_1 x}, e^{r_2 x}$ both satisfy $L(y) = 0$ in case $r_1 \neq r_2$. The case $r_1 = r_2$ is the double root case,

$$L = (D - r_1 I) \circ (D - r_1 I) = (D - r_1 I)^2 .$$

Exercise 1)

a) Let $f(x)$ be any differentiable function. Check that

$$(D - r_1 I)\left(f(x) e^{r_1 x}\right) = f'(x) e^{r_1 x} .$$

b) Deduce that

$$(D - r_1 I) e^{r_1 x} = 0$$

$$(D - r_1 I)^2 x\, e^{r_1 x} = (D - r_1 I) e^{r_1 x} = 0$$

$$(D - r_1 I)^3 x^2 e^{r_1 x} = 0$$

etc.

(1b explains the "magic" computation we did on Friday, as well as the more general case of repeated roots, on the next page.)

Here's the general algorithm for Case 2, repeated real roots: If

$$p(r) = (r - r_1)^{k_1} (r - r_2)^{k_2} \cdots (r - r_m)^{k_m}$$

then (as before) $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_m x}$ are independent solutions, but since $m < n$ there aren't enough of them to be a basis. Here's how you get the rest: For each $k_j > 1$, you actually get independent solutions

$$e^{r_j x}, x e^{r_j x}, x^2 e^{r_j x}, \ldots, x^{k_j - 1} e^{r_j x}.$$

This yields k_j solutions for each root r_j, so since $k_1 + k_2 + \ldots + k_m = n$ you get a total of n solutions to the differential equation. Exercise 1b explains why these are all solutions to $L(y) = 0$. Using the limiting method we discussed on Friday, it's not to hard to show that all n of these solutions are indeed linearly independent, because they all have different growth rates as $x \rightarrow \infty$. Thus they are in fact a basis for the solution space to $L(y) = 0$.

Exercise 2) Realizing that the function $e^{0x} = 1$, explicitly antidifferentiate to show that the solution space to the differential equation for $y(x)$

$$y^{(4)} - y^{(3)} = 0$$

agrees with what you would get using the repeated roots algorithm above. Hint: first find $v = y'''$, using $v' - v = 0$, then antidifferentiate three times to find y_H.

Case 3) $p(r)$ has some complex roots. The punch line is that complex exponential functions still work - but that rather than the complex exponential functions we use related real-valued functions that are products of exponential and trigonometry functions. To understand how this all comes about, we need to learn Euler's formula . This also lets us review some important Taylor's series facts from Calc 2. As it turns out, complex number arithmetic and complex exponential functions actually are important in many engineering and science applications.

Recall the Taylor-Maclaurin formula from Calculus

$$f(x) \sim f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + ... + \frac{1}{n!}f^{(n)}(0)x^n + ...$$

(Recall that the partial sum polynomial through order n, $p_n(x)$, matches f and its first n derivatives at $x_0 = 0$. When you studied Taylor series in Calculus you sometimes expanded about points other than $x_0 = 0$. You also needed error estimates to figure out on which intervals the Taylor polynomials actually coverged back to f.)

Exercise 3) Use the formula above to recall the three very important Taylor series for

3a) $e^x =$
3b) $\cos(x) =$
3c) $\sin(x) =$

Exercise 4) Let $x = i\,\theta$ and use the Taylor series for e^x as the definition of $e^{i\,\theta}$ in order to derive Euler's formula:

$$e^{i\,\theta} := \cos(\theta) + i\sin(\theta)\ .$$

From Euler's formula it makes sense to define

$$e^{a+b\,i} := e^a e^{b\,i} = e^a\left(\cos(b) + i\sin(b)\right)$$

for $a, b \in \mathbb{R}$. So for $x \in \mathbb{R}$ we also get

$$e^{(a+b\,i)x} = e^{a\,x}\left(\cos(b\,x) + i\sin(b\,x)\right) = e^{a\,x}\cos(b\,x) + i\,e^{a\,x}\sin(b\,x)\,.$$

For a complex function $f(x) + i\,g(x)$ we define the derivative by

$$D_x\left(f(x) + i\,g(x)\right) := f'(x) + i\,g'(x)\,.$$

It's straightforward to verify (but would take some time to check all of them) that the usual differentiation rules, i.e. sum rule, product rule, quotient rule, constant multiple rule, all hold for derivatives of complex functions. The following rule pertains directly to our discussion:

Exercise 5) Check that $D_x\left(e^{(a+b\,i)x}\right) = (a+b\,i)e^{(a+b\,i)x}$, i.e.

$$D_x\,e^{r\,x} = r\,e^{r\,x}$$

even if r is complex.

Now return to our differential equation questions, with

$$L = D^n + a_{n-1}D^{n-1} + \ldots. + a_1 D + a_0 I\,.$$

Then even for complex $r = a + b\,i$ $\left(a, b \in \mathbb{R}\right)$, our work above shows that

$$L\left(e^{r\,x}\right) = e^{r\,x}\left(r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0\right) = e^{r\,x}\,p(r)\,.$$

So if $r = a + b\,i$ is a complex root of $p(r)$ then $e^{r\,x}$ is a complex-valued function solution to $L(y) = 0$. But L is linear, and because of how we take derivatives of complex functions, we compute in this case that

$$0 + 0\,i = L\left(e^{r\,x}\right) = L\left(e^{a\,x}\cos(b\,x) + i\,e^{a\,x}\sin(b\,x)\right)$$

$$= \left(D^n + a_{n-1}D^{n-1} + \ldots. + a_1 D + a_0 I\right)\left(e^{a\,x}\cos(b\,x) + i\,e^{a\,x}\sin(b\,x)\right)$$

$$= L\left(e^{a\,x}\cos(b\,x)\right) + i\,L\left(e^{a\,x}\sin(b\,x)\right)\,.$$

Equating the real and imaginary parts in the first expression to those in the final expression (because that's what it means for complex numbers to be equal) we deduce

$$0 = L\left(e^{a\,x}\cos(b\,x)\right)$$

$$0 = L\left(e^{a\,x}\sin(b\,x)\right)\,.$$

Upshot: If $r = a + b\,i$ is a complex root of the characteristic polynomial $p(r)$ then

$$y_1 = e^{a\,x}\cos(b\,x)$$

$$y_2 = e^{a\,x}\sin(b\,x)$$

are two solutions to $L(y) = 0$. (The conjugate root $a - b\,i$ would give rise to y_1, $-y_2$, which have the same span.

Case 3) Let L have characteristic polynomial

$$p(r) = r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0$$

with real constant coefficients $a_{n-1}, \ldots, a_1, a_0$. If $(r - (a + b\,i))^k$ is a factor of $p(r)$ then so is the conjugate factor $(r - (a - b\,i))^k$. Associated to these two factors are $2\,k$ real and independent solutions to $L(y) = 0$, namely

$$\begin{array}{c} e^{a x}\cos(b x),\ e^{a x}\sin(b x) \\ x\,e^{a x}\cos(b x),\ x\,e^{a x}\sin(b x) \\ \vdots \qquad\qquad \vdots \\ x^{k-1} e^{a x}\cos(b x),\ x^{k-1} e^{a x}\sin(b x) \end{array}$$

Combining cases 1,2,3, yields a complete algorithm for finding the general solution to $L(y) = 0$, as long as you are able to figure out the factorization of the characteristic polynomial $p(r)$.

Exercise 6) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 9\,y = 0\,.$$

(You were told a basis in last week's hw....now you know where it came from.)

Exercise 7) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 6\,y' + 13\,y = 0\,.$$

(You were told a basis in last week's hw....now you know where it came from.)

Exercise 8) Suppose a 7^{th} order linear homogeneous DE has characteristic polynomial

$$p(r) = \left(r^2 + 6\,r + 13\right)^2 (r - 2)^3\,.$$

What is the general solution to the corresponding homogeneous DE?