Math 2250-4
Fri Mar 30

Today and Monday we'll discuss interesting applications of Laplace transforms to differential equations, using sections 10.4,10.5,EP7.6. The new table entries we'll focus on are

$f(t)$ with $\lvert f(t)\rvert \le C e^{M t}$	$F(s) := \int_0^\infty f(t) e^{-s t}\, dt$ for $s > M$	comments
$u(t-a)$ unit step function	$\frac{e^{-a s}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\, u(t-a)$	$e^{-a s} F(s)$	more complicated on/off
$\delta(t-a)$	$e^{-a s}$	unit impulse/delta "function"
$\int_0^t f(\tau) g(t-\tau)\, d\tau$	$F(s) G(s)$	convolution integrals to invert Laplace transform products
$f(t)$ with period p	$\frac{1}{1-e^{-ps}} \int_0^p f(t) e^{-s t}\, dt$	for periodic forcing which is not sinusoidal.

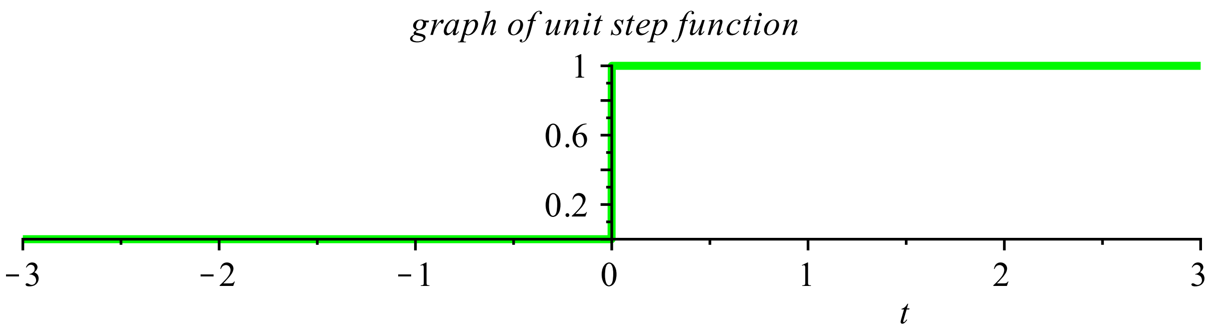
The unit step function with jump at $t = 0$ is defined to be

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases} .$$

Its graph is shown below. Notice that this function is called the "Heaviside" function in Maple, after the person who popularized it (among a lot of other accomplishments) and not because it's heavy on one side.

http://en.wikipedia.org/wiki/Oliver_Heaviside

```
> with(plots) :
  plot(Heaviside(t), t=-3 ..3, color = green, title = `graph of unit step function`);
```



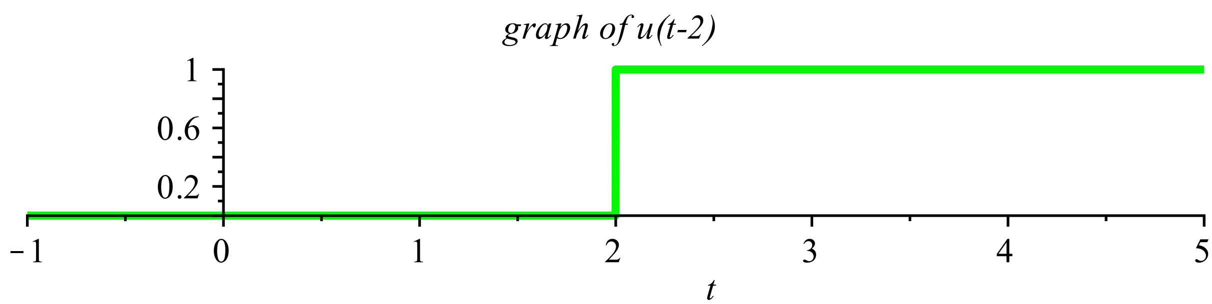
Notice that technically the vertical line should not be there - a more precise picture would have a solid point at $(0, 1)$ and a hollow circle at $(0, 0)$, for the graph of $u(t)$. In terms of Laplace transform it doesn't actually matter what we define $u(0)$ to be.

Then

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

and has graph that is a horizontal translation of the original graph, e.g. for $a = 2$:

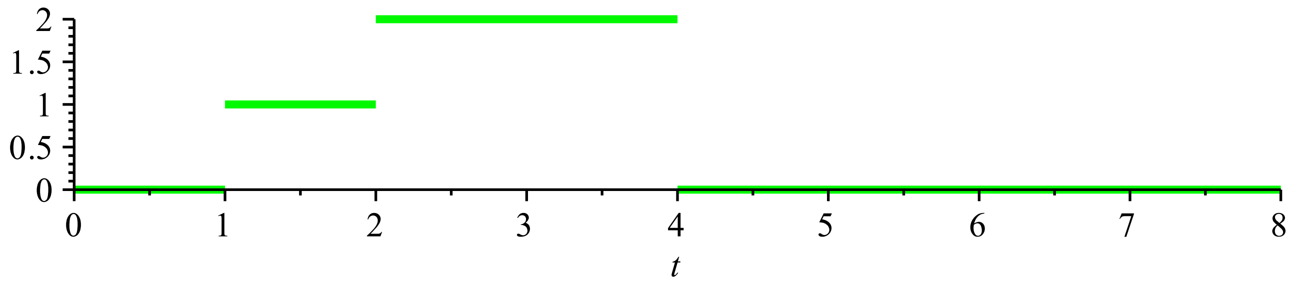
> plot(Heaviside(t − 2), t = −1 ..5, color = green, title = `graph of u(t-2)`);



Exercise 1) Verify the table entries

$u(t-a)$ unit step function	$\frac{e^{-a s}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\, u(t-a)$	$e^{-a s} F(s)$	more complicated on/off

Exercise 2) Consider the function $f(t)$ which is zero for $t > 4$ and with the following graph. Use linearity and the unit step function entry to compute the Laplace transform $F(s)$.




```
> plot(Heaviside(t − 1) + Heaviside(t − 2) − 2·Heaviside(t − 4), t = 0 ..8, color = green);
  with(inttrans) :
  laplace(Heaviside(t − 1) + Heaviside(t − 2) − 2·Heaviside(t − 4), t, s);
```

Setup: an under-employed mathematician/engineer/scientist (your choice)

likes to take his/her child to the swings...

recall pendulum (linearized) eqtn, without forcing, for $\theta = \theta(t)$

$L\theta'' + g\theta = 0$

$\dashrightarrow x'' + g\frac{x}{L} = 0$

$\dashrightarrow m x'' + \frac{mg}{L} x = F_0 \cos\omega t \leftarrow$ parent forcing (?!)

$x(t) = L \sin\theta(t)$

$\approx L\theta$ for small θ

so $x'' \approx L\theta''$

$\dashrightarrow x'' + \frac{g}{L} x = \frac{F_0}{m} \cos\omega t$

$x'' + \omega_0^2 x = \frac{F_0}{m} \cos\omega t$

for resonance $\omega = \omega_0$

construct swing with $L = g \approx 9.8$ m.

so $\omega_0^2 = 1$, $T_0 = 2\pi \approx 6.2$ seconds

parent pushes sinusoidally for exactly 5 cycles, and with $\frac{F_0}{m} = .2$ and then releases:

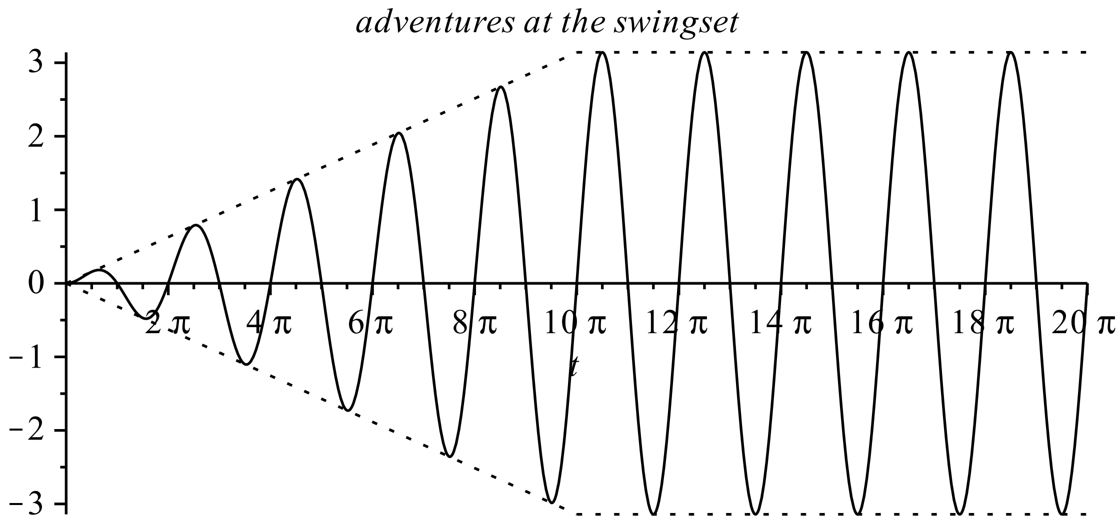
Exercise 3a) Explain why the description above leads to the differential equation initial value problem for $x(t)$

$$x''(t) + x(t) = .2\cos(t)\left(1 - u(t - 10\pi)\right)$$
$$x(0) = 0$$
$$x'(0) = 0$$

3b) Find $x(t)$. Show that after the parent stops pushing, the child is oscillating with an amplitude of exactly π meters.

Pictures for the swing:

```
> plot1 := plot(.1·t·sin(t), t = 0..10·Pi, color = black) :
  plot2 := plot(Pi·sin(t), t = 10·Pi..20·Pi, color = black) :
  plot3 := plot(Pi, t = 10·Pi..20·Pi, color = black, linestyle = 2) :
  plot4 := plot(-Pi, t = 10·Pi..20·Pi, color = black, linestyle = 2) :
  plot5 := plot(.1·t, t = 0..10·Pi, color = black, linestyle = 2) :
  plot6 := plot(-.1·t, t = 0..10·Pi, color = black, linestyle = 2) :
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title = `adventures at the swingset`);
```



Alternate approach via Chapter 5:

<u>step 1)</u> solve

$$x''(t) + x(t) = .2\cos(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

for $0 \le t \le 10\pi$.

<u>step 2)</u> Then solve

$$y''(t) + y(t) = 0$$
$$y(0) = x(10\pi)$$
$$y'(0) = x'(10\pi)$$

and set $x(t) = y(t - 10)$ for $t > 10$.

Laplace transform convolution

For Laplace transforms the convolution of *f* and *g* is defined to be

$$f*g(t) := \int_0^t f(\tau)g(t-\tau)\,d\tau\,.$$

Exercise 4) Show $f*g(t) = g*f(t)$ by doing a change of variables in the convolution integral.

$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	convolution integrals to invert Laplace transform products

Exercise 4) Verify that the convolution integral table entry is correct, for

$$f(t) = \sin(t) \quad F(s) = \frac{1}{s^2+1}$$

$$g(t) = \cos(t) \quad G(s) = \frac{s}{s^2+1}$$

$$f*g(t) \qquad F(s)G(s) = \frac{s}{\left(s^2+1\right)^2}\,.$$

Hint: you might use the trig identity

$$\sin(\tau)^2 = \frac{1-\cos(2\tau)}{2}\,.$$

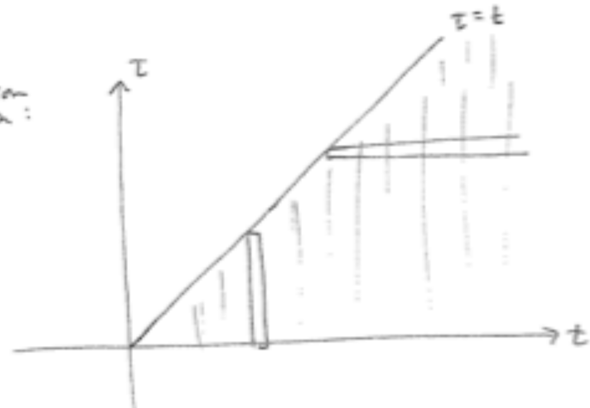
proof of convolution theorem:

(is a good review of iterated integrals)

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)\,g(t-\tau)\,d\tau\right) dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(\tau)\,g(t-\tau)\,d\tau\,dt$$

integration region:



interchange limits:

$$= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\,g(t-\tau)\,dt\,d\tau$$

$$= \int_0^\infty \int_\tau^\infty e^{-s\tau} f(\tau)\, e^{-s(t-\tau)} g(t-\tau)\,dt\,d\tau \qquad \text{(pattern recognition)}$$

$$= \int_0^\infty e^{-s\tau} f(\tau) \left[\int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\,dt \right] d\tau$$

$$\tilde{t} = t - \tau$$
$$d\tilde{t} = dt$$

$$\underbrace{\left[\int_0^\infty e^{-s\tilde{t}} g(\tilde{t})\,d\tilde{t} \right]}_{G(s)}$$

$$= G(s) \int_0^\infty e^{-s\tau} f(\tau)\,d\tau$$

$$= G(s)\,F(s) \qquad !!$$

$f(t)$, with $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$	↓ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1	$\frac{1}{s}$ $(s > 0)$	☐
t	$\frac{1}{s^2}$	☐
t^2	$\frac{2}{s^3}$	☐
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	☐
$e^{\alpha t}$	$\frac{1}{s-\alpha}$ $(s > \Re(a))$	☐
$\cos(kt)$	$\frac{s}{s^2+k^2}$ $(s > 0)$	☐
$\sin(kt)$	$\frac{k}{s^2+k^2}$ $(s > 0)$	☐
$\cosh(kt)$	$\frac{s}{s^2-k^2}$ $(s > k)$	☐
$\sinh(kt)$	$\frac{k}{s^2-k^2}$ $(s > k)$	☐
$e^{at}\cos(kt)$	$\frac{(s-a)}{(s-a)^2+k^2}$ $(s > a)$	☐
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$ $(s > a)$	☐
$e^{at}f(t)$	$F(s-a)$	☐
$u(t-a)$	$\frac{e^{-as}}{s}$	
$f(t-a)\,u(t-a)$	$e^{-as}F(s)$	
$\delta(t-a)$	e^{-as}	
$f'(t)$	$s\,F(s) - f(0)$	☐
$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$	☐
$f^{(n)}(t),\ n \in \mathbb{N}$	$s^n F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	☐
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	☐
$t\,f(t)$	$-F'(s)$	☐
$t^2 f(t)$	$F''(s)$	☐
$t^n f(t),\ n \in \mathbb{Z}$	$(-1)^n F^{(n)}(s)$	☐
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	
$t\cos(kt)$	$\frac{s^2-k^2}{(s^2+k^2)^2}$	☐

$\frac{1}{2k} t \sin(kt)$ $\frac{1}{2k^3}(\sin(kt) - kt\cos(kt))$ $t e^{at}$ $t^n e^{at}, n \in \mathbb{Z}$	$\frac{s}{(s^2+k^2)^2}$ $\frac{1}{(s^2+k^2)^2}$ $\frac{1}{(s-a)^2}$ $\frac{n!}{(s-a)^{n+1}}$	☐ ☐ ☐ ☐
$\int_0^t f(\tau)g(t-\tau)\, d\tau$	$F(s)G(s)$	
$f(t)$ with period p	$\frac{1}{1-e^{-ps}}\int_0^p f(t)e^{-st}\, dt$	

Laplace transform table