

Fri Mar 23

10.1-10.2 Laplace transform, and application to DE IVPs, including Chapter 5.

- This week's hw is due Wed Mar 28, because our second midterm exam is Thur Mar 29. The hw covers sections 10.1-10.3, and you are also responsible for these sections on the exam.
- The Laplace transform is a linear transformation " \mathcal{L} " that converts piecewise continuous functions $f(t)$, defined for $t \geq 0$ and with at most exponential growth ($|f(t)| \leq Ce^{Mt}$ for some values of C and M), into functions $F(s)$ defined by

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^{\infty} f(t)e^{-st} dt.$$

- Notice that the integral formula for $F(s)$ is only defined for sufficiently large s , and certainly for $s > M$, because as soon as $s > M$ the integrand is decaying exponentially, so the improper integral converges.
- The convention is to use lower case letters for the input functions and (the same) capital letters for their Laplace transforms, as we did for $f(t)$ and $F(s)$ above. Thus if we called the input function $g(t)$ then we would denote the Laplace transform by $G(s)$.

Taking Laplace transforms seems like a strange thing to do. And yet, the Laplace transform \mathcal{L} is just one example of a collection of useful "integral transforms". \mathcal{L} is especially good for solving IVPs for linear DEs, as we shall see starting today. Other famous transforms - e.g. Fourier series and Fourier transform are extremely important in studying linear partial differential equations, as you will see in e.g. Math 3150.

Exercise 1) Use the definition of Laplace transform

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^{\infty} f(t)e^{-st} dt$$

to check the following facts, which you will also find inside the front cover of your text book.

a) $\mathcal{L}\{1\}(s) = \frac{1}{s} \quad (s > 0)$

b) $\mathcal{L}\{e^{\alpha t}\}(s) = \frac{1}{s - \alpha} \quad (s > \alpha \text{ if } \alpha \in \mathbb{R}, s > a \text{ if } \alpha = a + ki \in \mathbb{C})$

c) Laplace transform is linear, i.e.

$$\begin{aligned} \mathcal{L}\{f_1(t) + f_2(t)\}(s) &= F_1(s) + F_2(s). \\ \mathcal{L}\{cf(t)\}(s) &= cF(s). \end{aligned}$$

d) Use linearity and your work above to compute $\mathcal{L}\{3 - 4e^{-2t}\}(s)$.

For the DE's like we've just been studying in Chapter 5, the following Laplace transforms are very important:

Exercise 2) Use complex number algebra, including Euler's formula, linearity, and the result from 1b that

$$\mathcal{L}\{e^{(a + ki)t}\}(s) = \frac{1}{s - (a + ki)}$$

to verify that

$$\underline{a)} \quad \mathcal{L}\{\cos(kt)\}(s) = \frac{s}{s^2 + k^2}$$

$$\underline{b)} \quad \mathcal{L}\{\sin(kt)\}(s) = \frac{k}{s^2 + k^2}$$

$$\underline{c)} \quad \mathcal{L}\{e^{at}\cos(kt)\}(s) = \frac{s - a}{(s - a)^2 + k^2}$$

$$\underline{d)} \quad \mathcal{L}\{e^{at}\sin(kt)\}(s) = \frac{k}{(s - a)^2 + k^2} .$$

(Notice that if we tried doing these Laplace transforms directly from the definition, the integrals would be messy but we could attack them via integration by parts or integral tables.)

It's a theorem (hard to prove but true) that a given Laplace transform $F(s)$ can arise from at most one piecewise continuous function $f(t)$. (Well, except that the values of f at the points of discontinuity can be arbitrary, as they don't affect the integral used to compute $F(s)$.) Therefore you can read Laplace transform tables in either direction, i.e. not only to deduce Laplace transforms, but inverse Laplace transforms $\mathcal{L}^{-1}\{F(s)\}(t) = f(t)$ as well.

Exercise 3) Use the Laplace transforms we've computed and linearity to compute

$$\mathcal{L}^{-1}\left\{\frac{7}{s} + \frac{1}{s^2 + 16} - \frac{10s}{s^2 + 16}\right\}(t) .$$

$f(t)$ $ f(t) \leq Ce^{Mt}$	$F(s) := \int_0^{\infty} f(t)e^{-st} dt$ for $s > M$
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$
1	$\frac{1}{s} \quad (s > 0)$
$e^{\alpha t}$	$\frac{1}{s - \alpha} \quad (s > \Re(\alpha))$
$\cos(kt)$	$\frac{s}{s^2 + k^2} \quad (s > 0)$
$\sin(kt)$	$\frac{k}{s^2 + k^2} \quad (s > 0)$
$e^{at}\cos(kt)$	$\frac{(s - a)}{(s - a)^2 + k^2} \quad (s > a)$
$e^{at}\sin(kt)$	$\frac{k}{(s - a)^2 + k^2} \quad (s > a)$
$f'(t)$	$s F(s) - f(0)$
$f''(t)$	$s^2 F(s) - s f(0) - f'(0)$

Laplace transform table

The integral transforms of DE's and PDE's were designed to have the property that they convert the corresponding linear DE and PDE problems into algebra problems. For the Laplace transform it's because of these facts:

Exercise 4a) Use integration by parts and the definition of Laplace transform to show that

$$\mathcal{L}\{f'(t)\}(s) = s \mathcal{L}\{f(t)\}(s) - f(0) = s F(s) - f(0) .$$

4b) Use the result of a, applied to the function $f'(t)$ to show that

$$\mathcal{L}\{f''(t)\}(s) = s^2 F(s) - s f(0) - f'(0) .$$

4c) What would you guess is the Laplace transform of $f'''(t)$? Could you check this?

Here's an example of using Laplace transforms to solve DE IVPs, in the context of Chapter 5 and the mechanical (and electrical) application problems we considered there.

Exercise 5) Consider the undamped forced oscillation IVP

$$\begin{aligned} x''(t) + 4x(t) &= 10 \cos(3t) \\ x(0) &= 2 \\ x'(0) &= 1 \end{aligned}$$

If $x(t)$ is the solution, then both sides of the DE are equal. Thus the Laplace transforms are equal as well...so, take the Laplace transform of each side and use algebra to find $\mathcal{L}\{x(t)\}(s) = X(s)$. Notice you've computed $X(s)$ without actually knowing $x(t)$! If you were happy to stay in "Laplace land" you'd be done. In any case, at this point you can use our table entries to find $x(t) = \mathcal{L}^{-1}\{X(s)\}$.

(Notice that if your algebra skills are good you've avoided having to use the Chapter 5 algorithm of (i) find x_H (ii) find an x_P (iii) $x = x_P + x_H$ (iv) solve IVP.) Magic!

$\left[\begin{array}{l} \text{> with(DEtools) :} \\ \text{> dsolve}(\{x''(t) + 4 \cdot x(t) = 10 \cdot \cos(3 \cdot t), x(0) = 2, x'(0) = 1\}); \# \text{ to check} \end{array} \right.$

Exercise 6) Use Laplace transform as above, to solve the IVP for the following underdamped, unforced oscillator DE:

$$\begin{aligned} x''(t) + 6x'(t) + 34x(t) &= 0 \\ x(0) &= 3 \\ x'(0) &= 1 \end{aligned}$$

$\left[\begin{array}{l} \text{> dsolve}(\{x''(t) + 6 \cdot x'(t) + 34 \cdot x(t) = 0, x(0) = 3, x'(0) = 1\}); \# \text{ to check} \\ \text{> } \end{array} \right.$