Math 2250-4
Tues Mar 20

5.5-5.6 Finding y_P for non-homogeneous linear differential equations, and applications to forced mechanical systems.

• Continue working through the difference cases of forced mechanical systems, in Monday's notes. We used our knowledge of the method of undetermined coefficients to understand the form of particular and therefore also general solutions to these non-homogeneous differential equations, and now we're working through examples of the various cases in more detail.

• We'll use Exercise 4 (the resonance example) in Monday's notes as an excuse to discuss the variation of parameters method for finding particular solutions, which is a more general method than undetermined coefficients (Exercise 1, below).

• After we finish Monday's notes for undamped forced oscillations, we'll use today's notes as an outline to discuss damped forced oscillations.

• We are discussing resonance and practical resonance phenomena in the simplest possible settings. Here are some links which address how these phenomena arise, also in more complicated real-world applications in which the dynamical systems are more complex and have more components. Our baseline cases are the starting points for understanding these more complicated systems. We'll also address some more complicated applications when we move on to systems of differential equations.

http://en.wikipedia.org/wiki/Mechanical_resonance (wikipedia page with links)
http://www.nset.org.np/nset/php/pubaware_shaketable.php (shake tables for earthquake modeling)
http://www.youtube.com/watch?v=M_x2jOKAhZM (an engineering class demo shake table)
http://www.youtube.com/watch?v=j-zczJXSxnw (Tacoma narrows bridge)
http://en.wikipedia.org/wiki/Electrical_resonance (wikipedia page with links)
http://en.wikipedia.org/wiki/Crystal_oscillator (crystal oscillators)

Exercise 1) Use the variation of parameters method to find a particular solution related to yesterday's Exercise 4 - in that example we used a general solution in which a particular solution was found using the method of undetermined coefficients. For computational simplicity consider the specific differential equation

$$x'' + 9x = 80\cos(3t)$$

rather than the more general case

$$x'' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega_0 t) .$$

Damped forced oscillations $(c > 0)$ for $x(t)$:

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\,t)$$

Undetermined coefficients for $x_P(t)$:

$$\begin{aligned} k\,&\left[x_P = A\cos(\omega\,t) + B\sin(\omega\,t)\right] \\ +\,c\,&\left[x_P' = -A\,\omega\sin(\omega\,t) + B\,\omega\cos(\omega\,t)\right] \\ +\,m\,&\left[x_P'' = -A\,\omega^2\cos(\omega\,t) - B\,\omega^2\sin(\omega\,t)\right]\ . \end{aligned}$$

$$\begin{aligned} L(x_P) &= \cos(\omega\,t)\left(k\,A + c\,B\,\omega - m\,A\,\omega^2\right) \\ &\quad + \sin(\omega\,t)\left(k\,B - c\,A\,\omega - m\,B\,\omega^2\right)\ . \end{aligned}$$

Collecting and equating coefficients yields the matrix system

$$\begin{bmatrix} k - m\,\omega^2 & c\,\omega \\ -c\,\omega & k - m\,\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix},$$

which has solution

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2} \begin{bmatrix} k - m\,\omega^2 & -c\,\omega \\ c\,\omega & k - m\,\omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \frac{F_0}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2} \begin{bmatrix} k - m\,\omega^2 \\ c\,\omega \end{bmatrix}$$

In phase-amplitude form this reads

$$x_P = A\cos(\omega\,t) + B\sin(\omega\,t) = C\cos(\omega\,t - \alpha)$$

with

$$C = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}\ . \quad \text{(Check!)}$$

$$\cos(\alpha) = \frac{k - m\,\omega^2}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}\ .$$

And the general solution $x(t) = x_P(t) + x_H(t)$ is given by

- underdamped: $x = x_{sp} + x_{tr} = C\cos(\omega\,t - \alpha) + e^{-p\,t}C_1\cos(\omega_1 t)$.
- critically-damped: $x = x_{sp} + x_{tr} = C\cos(\omega\,t - \alpha) + e^{-p\,t}\left(c_1 t + c_2\right)$.
- over-damped: $x = x_{sp} + x_{tr} = C\cos(\omega\,t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$.

Important to note:

- The amplitude C in x_{sp} can be quite large relative to $\frac{F_0}{m}$ if $\omega \approx \omega_0$ and $c \approx 0$, because the denominator will then be close to zero. This phenomenon is practical resonance.
- The phase angle α is always in the first or second quadrant.

Exercise 2) Solve the IVP for $x(t)$:

$$x'' + 2\,x' + 26\,x = 82\cos(4\,t)$$
$$x(0) = 6$$
$$x'(0) = 0\,.$$

Solution:

$$x(t) = \sqrt{41}\cos(4\,t - \alpha) + \sqrt{10}\,e^{-t}\cos(5\,t - \beta)$$
$$\alpha = \arctan(0.8),\ \beta = \arctan(-3)\,.$$

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> with(DEtools) :
> dsolve({x''(t) + 2·x'(t) + 26·x(t) = 82·cos(4·t), x(0) = 6, x'(0) = 0});
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$$x(t) = -3\,e^{-t}\sin(5\,t) + e^{-t}\cos(5\,t) + 5\cos(4\,t) + 4\sin(4\,t) \qquad \textbf{(1)}$$

Practical resonance: The steady periodic amplitude C for damped forced oscillations (page 2) is

$$C(\omega) = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} \,.$$

Notice that as $\omega \to 0$, $C(\omega) \to \frac{F_0}{k}$ and that as $\omega \to \infty$, $C(\omega) \to 0$. The precise definition of practical resonance occuring is that $C(\omega)$ have a global maximum greater than $\frac{F_0}{k}$, on the interval $0 < \omega < \infty$.

Exercise 3a) Compute $C(\omega)$ for the damped forced oscillator equation related to the previous exercise, except with varying damping coefficient c:

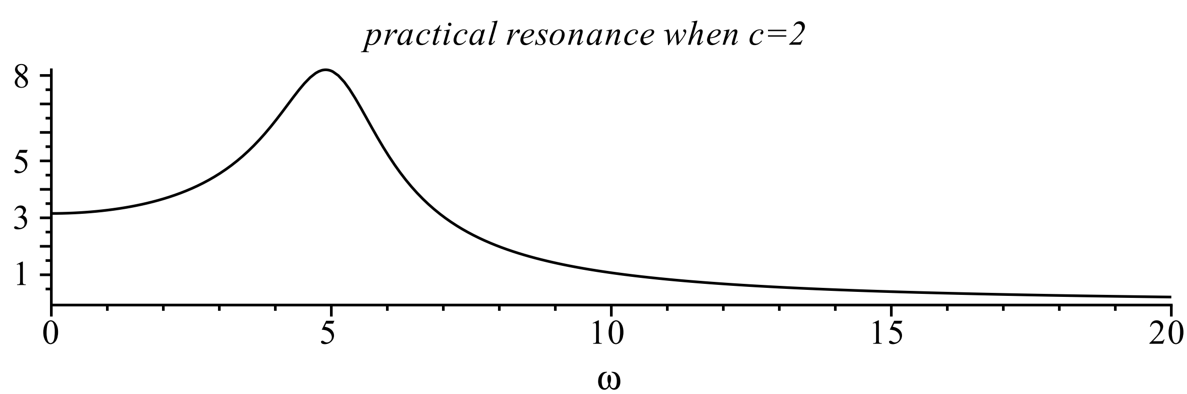
$$x'' + c\,x' + 26\,x = 82\cos(\omega\,t).$$

3b) Investigate practical resonance graphically, for $c = 2$ and for some other values as well. How would you check practical resonance analytically?

> $with(plots):$

> $C := (\omega, c) \to \frac{82}{\sqrt{\left(26 - \omega^2\right)^2 + c^2 \cdot \omega^2}}:$

> $plot(C(\omega, 2), \omega = 0..20, color = black, title = \text{`practical resonance when c=2`});$



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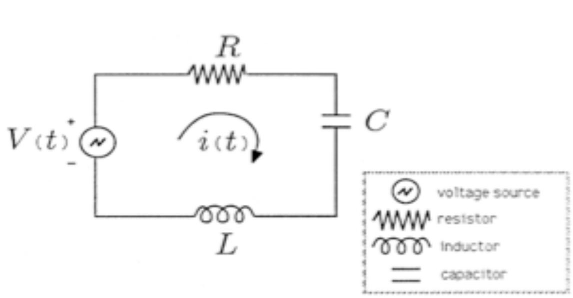
<u>The mechanical-electrical analogy, continued:</u> Practical resonance is usually bad in mechanical systems, but good in electrical circuits when signal amplification is a goal....recall from earlier in the course:



circuit element	voltage drop	units
inductor	$L\, I'(t)$	L Henries (H)
resistor	$R\, I(t)$	R Ohms (Ω)
capacitor	$\frac{1}{C} Q(t)$	C Farads (F)

http://cnx.org/content/m21475/latest/pic012.png

<u>Kirchoff's Law</u>: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts).

For $Q(t)$: $\quad L\, Q''(t) + R\, Q'(t) + \frac{1}{C} Q(t) = V(t) = E_0 \sin(\omega t)$

For $I(t)$: $\quad L\, I''(t) + R\, I'(t) + \frac{1}{C} I(t) = V'(t) = E_0\, \omega \cos(\omega t)$.

Transcribe the work on steady periodic solutions from the preceding pages! The general solution for $I(t)$ is

$$I(t) = I_{sp}(t) + I_{tr}(t) \,.$$

$$I_{sp}(t) = I_0 \cos(\omega t - \alpha) = I_0 \sin(\omega t - \gamma)\ , \quad \gamma = \alpha - \frac{\pi}{2} \,.$$

$$C(\omega) = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} \quad \Rightarrow I_0(\omega) = \frac{E_0\omega}{\sqrt{\left(\frac{1}{C} - L\,\omega^2\right)^2 + R^2\,\omega^2}}$$

$$\Rightarrow I_0(\omega) = \frac{E_0}{\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}} \,.$$

The denominator $\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}$ of $I_0(\omega)$ is called the i<u>mpedance</u> $Z(\omega)$ of the circuit (because the larger the impedance, the smaller the amplitude of the steady-periodic current that flows through the circuit). Notice that for fixed resistance, the impedance is minimized and the steady periodic current amplitude is maximized when

$\frac{1}{C\,\omega} = L\,\omega$, i.e.

$$C = \frac{1}{L\,\omega^2} \quad \text{if } L \text{ is fixed and } C \text{ is adjustable (old radios).}$$

$$L = \frac{1}{C\,\omega^2} \quad \text{if } C \text{ is fixed and } L \text{ is adjustable (like in today's demonstration!)}$$