Math 2250-4
Fri Mar 2

5.2-5.3 How to test functions for linear independence and how to find the solution space for n^{th} order linear homogeneous DE's with constant coefficients.

We know that for non-homogeneous linear DE's, the general solution $y = y_P + y_H$ is the sum of any single particular solution with the general solution to the homogeneous DE. The topic in section 5.3 is how to find y_H for any <u>constant coefficient</u> linear homogeneous differential equation

$$y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

(i.e. $a_{n-1}, a_{n-2}, ..., a_1, a_0$ are constants instead of more complicated functions of x).

Before we get to these algorithms for y_H let's expand on our discussion so far, about how to check whether collections of functions are linearly independent on specified intervals....this will be important because when it comes time to check that we've got bases of n functions for the solutions spaces above, if we verify linear independence, then span (and therefore basis) will follow.

<u>Exercise 0)</u> On Monday we rushed through the vector space fact that explains why n <u>independent vectors</u> (functions) in an <u>n-dimensional</u> space also automatically <u>span</u> the space. Take a minute to remind ourselves of this fact and why its true.

<u>Ways to check whether functions</u> $y_1, y_2, ... y_n$ are linearly independent on an interval:
In all cases you begin by writing the linear combination equation

$$c_1 y_1 + c_2 y_2 + ... + c_n y_n = 0$$

which is assumed to hold for all $x \in I$.

<u>Method 1)</u> Plug in different $x-$ values to get a system of algebraic equations for $c_1, c_2 .. c_n$. Either you'll get enough "different" equations to conclude that $c_1 = c_2 = ... = c_n = 0$, or you'll find a likely dependency.

<u>Exercise 1)</u> Use method 1 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1, \; y_2(x) = x, \; y_3(x) = x^2$$

are linearly independent. For example, try the system you get by plugging in $x = 0, -1, 1$ into the equation

$$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$$

Method 2) If your interval stretches to $+\infty$ or to $-\infty$ and your functions grow at different rates, you may be able to take limits (after dividing the dependency equation by appropriate functions of x), to deduce independence.

Exercise 2) Use method 2 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = x^2$$

are linearly independent. Hint: first divide the dependency equation by the fastest growing function, then let $x \to \infty$.

Method 3) If

$$c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0$$

$\forall\ x \in I$, then we can take derivatives to get a system

$$\begin{aligned} c_1 y_1 + c_2 y_2 + \ldots + c_n y_n &= 0 \\ c_1 y_1{}' + c_2 y_2{}' + \ldots + c_n y_n{}' &= 0 \\ c_1 y_1{}'' + c_2 y_2{}'' + \ldots + c_n y_n{}'' &= 0 \\ \\ c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \ldots + c_n y_n^{(n-1)} &= 0 \end{aligned}$$

(We could keep going, but stopping here gives us n equations in n unknowns.)
Plugging in any value of x_0 yields a homogeneous algebraic linear system of n equations in n unknowns, which is equivalent to the Wronskian matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1{}'(x_0) & y_2{}'(x_0) & \cdots & y'_n(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If this Wronskian matrix is invertible at even a single point $x_0 \in I$, then the functions are linearly independent! (So if the determinant is zero at even a single point $x_0 \in I$, then the functions are independent....strangely, even if the determinant was zero for all $x \in I$, then it could still be true that the functions are independent....but that won't happen if our n functions are all solutions to the same n^{th} order linear homogeneous DE.)

Exercise 3) Use method 3 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = x^2$$

are linearly independent.

Remark 1) Method 3 is usually not the easiest way to prove independence. But we like it when studying differential equations because (as we saw on Wednesday) the Wronskian matrix shows up when you're trying to solve initial value problems using linear combinations $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ of solutions $y_1, y_2, \ldots y_n$ to

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0.$$

As we showed, if the initial conditions for this homogeneous DE are

$$y(x_0) = b_0, y'(x_0) = b_1, \ldots y^{(n-1)}(x_0) = b_{n-1}$$

then you need to solve

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1{}'(x_0) & y_2{}'(x_0) & \cdots & y'_n(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}.$$

And so if you're using the Wronskian matrix method, and the matrix is invertible at x_0 then you are effectively directly checking that $y_1, y_2, \ldots y_n$ are a basis for the solution space, and you are ready to solve any initial value problem you want.

(But be careful: If you're solving a non-homogeneous differential equation then the solution is $y = y_P + y_H$ then the matrix equation to find the linear combination coefficients has the same Wronskian matrix but includes another vector in the equation - in fact it's the initial value vector for y_P. Can you see where it goes in the equation above?)

Remark 2) There is a seemingly magic consequence in the situation above, in which $y_1, y_2, \ldots y_n$ are all solutions to the same n^{th}-order homogeneous DE

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

(even if the coefficients aren't constants): If the Wronskian matrix of your solutions $y_1, y_2, \ldots y_n$ is invertible at a single point x_0, then $y_1, y_2, \ldots y_n$ are a basis because linear combinations solve all IVP's at x_0. But since they're a basis, that also means that linear combinations of $y_1, y_2, \ldots y_n$ solve all IVP's at any other point x_1. This is only possible if the Wronskian matrix at x_1 also reduces to the identity matrix at x_1 and so is invertible there too. In other words, the Wronskian determinant will either be non-zero $\forall\, x \in I$, or zero $\forall\, x \in I$, when your functions $y_1, y_2, \ldots y_n$ all happen to be solutions to the same n^{th} order homogeneous linear DE as above.

Exercise 4) Verify that $y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = x^2$ all solve the third order linear homogeneous DE

$$y''' = 0,$$

and that their Wronskian determinant is indeed non-zero $\forall\, x \in \mathbb{R}$.

Algorithms for the basis and general solution to

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

when the coefficients $a_{n-1}, a_{n-2}, ... a_1, a_0$ are all constant.

step 1) Try to find a basis made of exponential functions....try $y(x) = e^{r x}$. In this case

$$L(y) = e^{r x}\left(r^n + a_{n-1}r^{n-1} + ... + a_1 r + a_0\right) = e^{r x} p(r) .$$

We call this polynomial $p(r)$ the characteristic polynomial for the differential equation, and can read off what it is directly from the expression for $L(y)$. For each root r_j of $p(r)$, we get a solution $e^{r_j x}$ to the homogeneous DE.

Case 1) If $p(r)$ has n distinct (i.e. different) real roots $r_1, r_2, ..., r_n$, then

$$e^{r_1 x}, e^{r_2 x}, ... , e^{r_n x}$$

is a basis for the solution space; i.e. the general solution is given by

$$y_H(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + ... + c_n e^{r_n x} .$$

Exercise 5) By construction, $e^{r_1 x}, e^{r_2 x}, ... , e^{r_n x}$ all solve the differential equation. Show that they're linearly independent. This will be enough to verify that they're a basis for the solution space, since we know the solution space is n-dimensional. Hint: The easiest way to show this is to list your roots so that $r_1 < r_2 < ... < r_n$ and to use a limiting argument....in your homework for today and in class we've seen specific examples like this, and used the Wronskian matrix and Wronskian (determinant) to check independence.

Case 2) Repeated real roots. In this case $p(r)$ has all real roots $r_1, r_2, \ldots r_m$ $(m < n)$ with the r_j all different, but some of the factors $(r - r_j)$ in $p(r)$ appear with powers bigger than 1 . In other words, $p(r)$ factors as

$$p(r) = (r - r_1)^{k_1} (r - r_2)^{k_2} \cdots (r - r_m)^{k_m}$$

with some of the $k_j > 1$, and $k_1 + k_2 + \ldots + k_m = n$.

Start with a small example: The case of a second order DE for which the characteristic polynomial has a double root.

Exercise 6) Consider

$$L(y) := y'' + a_1 y' + a_0 y$$

with $p(r) = r^2 + a_1 r + a_0 = (r - r_1)^2$. Show that $e^{r_1 x}, x e^{r_1 x}$ are a basis for the solution space to $L(y) = 0$, so the general homogeneous solution is $y_H(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$. Start by checking that $x e^{r_1 x}$ actually solves the DE.

Here's the general algorithm: If

$$p(r) = (r - r_1)^{k_1}(r - r_2)^{k_2} \cdots (r - r_m)^{k_m}$$

then (as before) $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_m x}$ are independent solutions, but since $m < n$ there aren't enough of them to be a basis. Here's how you get the rest: For each $k_j > 1$, you actually get independent solutions

$$e^{r_j x}, x\,e^{r_j x}, x^2 e^{r_j x}, \ldots, x^{k_j - 1} e^{r_j x}.$$

This yields k_j solutions for each root r_j, so since $k_1 + k_2 + \ldots + k_m = n$ you get a total of n solutions to the differential equation. There's a good explanation in the text as to why these additional functions actually do solve the differential equation, see pages 316-318 and the discussion of "polynomial differential operators". Using the limiting method we discussed earlier, it's not to hard to show that all n of these solutions are indeed linearly independent, so they are in fact a basis for the solution space to $L(y) = 0$.

Exercise 7) Realizing that the function $e^{0\,x} = 1$, explicitly antidifferentiate to show that the solution space to the differential equation for $y(x)$

$$y^{(4)} - y^{(3)} = 0$$

agrees with what you would get using the repeated roots algorithm in Case 2 above. Hint: first find $v = y'''$, using $v' - v = 0$, then antidifferentiate three times to find y_H

Case 3) Complex roots - this will be Monday fun!