

Math 2250-4

Wed Jan 25

2.2 Autonomous differential equations and applications

2.3 Improved velocity models.

Final intro to Maple session in LCB 115:

Wednesday 3-3:50 p.m.

Exercise 1 (vocabulary): Yesterday we discussed *equilibrium solutions* and *stability* for *autonomous first order differential equations*. What do these words mean?

a) autonomous first order differential equation:

b) equilibrium solution to autonomous DE.

c) stable equilibrium solution.

d) unstable equilibrium solution.

e) asymptotically stable equilibrium solution.

now go finish the discussion on last pages of Tuesday's notes... did anyone think of an example of an autonomous first order DE for which there is a stable equilibrium solution that is not also asymptotically stable? We might play with dfield too...

Application: harvesting a logistic population...text p.97 (or, why do fisheries sometimes seem to die out "suddenly"?) Consider the DE

$$P'(t) = aP - bP^2 - h.$$

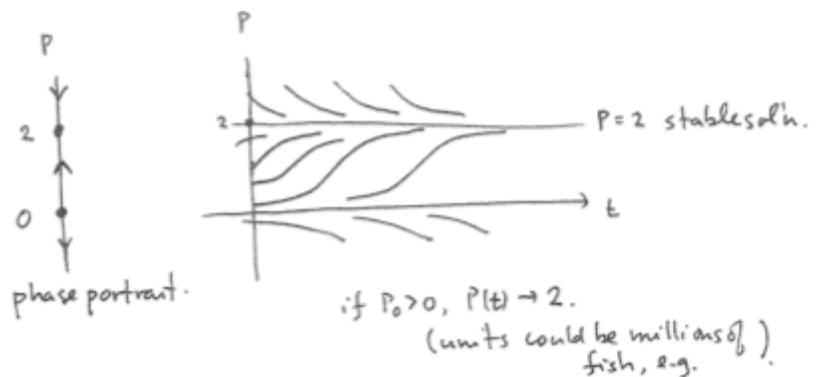
Notice that the first two terms represent a logistic rate of change, but we are now harvesting the population at a rate of h units per time. For simplicity we'll assume we're harvesting fish per year (or thousands of fish per year etc.) One could model different situations, e.g. constant "effort" harvesting, in which the effect on how fast the population was changing could be hP instead of P .

For computational ease we will assume $a = 2$, $b = 1$. (One could actually change units of population and time to reduce to this case.)

for computational simplicity
take $a = 2$, $b = 1$

Case 0 no harvesting

$$P'(t) = 2P - P^2 \\ = P(2 - P)$$



with harvesting:

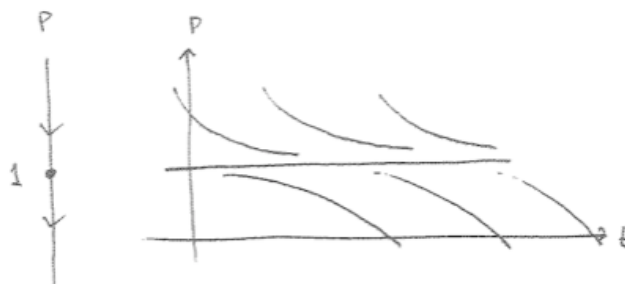
$$P'(t) = 2P - P^2 - h \\ = -(P^2 - 2P + h) \\ = -(P - P_1)(P - P_2) \\ P_1, P_2 = \frac{2 \pm \sqrt{4 - 4h}}{2} \\ = 1 \pm \sqrt{1 - h}$$

Case 1: subcritical harvesting
 $0 < h < 1$



Case 2. Critical harvesting
 $h = 1$

$$P'(t) = -(P-1)^2$$



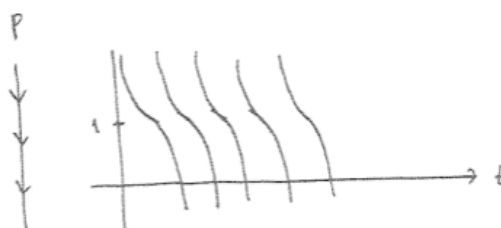
Case 3 Over harvesting
 $h > 1$

complex roots.

$$P'(t) = -(P^2 - 2P + h)$$

$$= -[(P-1)^2 + (h-1)]$$

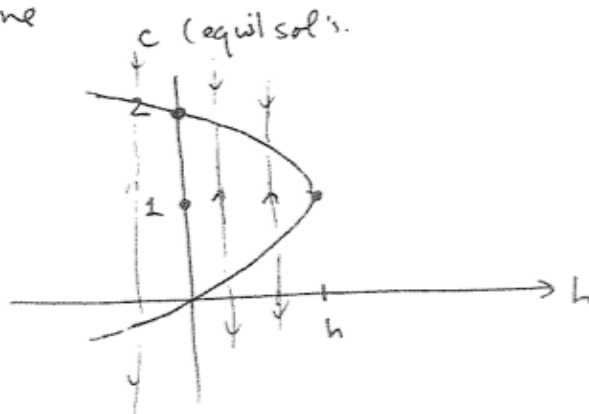
$$< 0.$$



This model gives a plausible explanation for why many fisheries have "unexpectedly" collapsed in modern history. If $h < 1$ but near 1 and something perturbs the system a little bit (a bad winter, or a slight increase in fishing pressure), then the population and/or model could suddenly shift so that $P(t) \rightarrow 0$ very quickly.

Here's one picture that summarizes all the cases - you can think of it as collection of the phase diagrams for different fishing pressures h . The upper half of the parabola represents the stable equilibria, and the lower half represents the unstable equilibria. Diagrams like this are called "bifurcation diagrams". In the sketch below, the point on the h -axis should be labeled $h = 1$, not h . What's shown is the parabola of equilibrium solutions, $c = 1 \pm \sqrt{1-h}$, i.e. $2c - c^2 - h = 0$, i.e. $h = c(2-c)$.

"bifurcation diagram" of
 equilibrium solutions in the h - c
 plane



Begin 2.3: improved velocity models.

For particle motion along a line, with

$$\begin{aligned} &\text{position } x(t) \text{ (or } y(t) \text{) ,} \\ &\text{velocity } x'(t) = v(t) \text{ , and} \\ &\text{acceleration } x''(t) = v'(t) = a(t) \end{aligned}$$

We have Newton's 2nd law

$$m v'(t) = F$$

where F is the net force.

- We're very familiar with constant force $F = m \alpha$:

$$\begin{aligned} v'(t) &= \alpha \\ v(t) &= \alpha t + v_0 \\ x(t) &= \frac{1}{2} \alpha t^2 + v_0 t + x_0 . \end{aligned}$$

Examples we've seen a lot of:

- $\alpha = -g$ near the surface of the earth, if up is the positive direction, or $\alpha = g$ if down is the positive direction.
- boats or cars subject to constant acceleration or deceleration.

New today !!! Try to account for frictional drag forces.

$$m v'(t) = m \alpha + F_f$$

Empirically/mathematically the drag forces F_f depend on velocity, in such a way that their magnitude is

$$|F_f| \approx k |v|^p , \quad 1 \leq p \leq 2 .$$

- $p = 1$ (linear model, drag proportional to velocity):

$$m v'(t) = m \alpha - k v$$

This linear model makes sense for "slow" velocities, as a linearization of the frictional force function, assuming it's differentiable...recall Taylor series:

$$F_f(v) = F_f(0) + F_f'(0) v + \frac{1}{2!} F_f''(0) v^2 + \dots$$

$F_f(0) = 0$ and for small v the higher order terms are negligible, so

$$F_f(v) \approx F_f'(0) v \approx -k v$$

since the frictional force opposes the direction of motion, so has the opposite sign of velocity.

Exercise 2a: Rewrite the linear drag model as

$$v'(t) = \alpha - \rho v$$

where the $\rho = \frac{k}{m}$. Construct the phase diagram for v . Notice that $v(t)$ has exactly one constant (equilibrium) solution, and find it. Its value is called the *terminal velocity*. Explain why *terminal velocity* is an appropriate term of art, based on your phase diagram.

2b) Solve the IVP

$$\begin{aligned} v'(t) &= \alpha - \rho v \\ v(0) &= v_0 \end{aligned}$$

and verify your phase diagram analysis.

• $p = 2$, for the power in the frictional force. Accounting for the fact that friction opposes direction of motion we get

$$\begin{aligned} m v'(t) &= m \alpha - k v^2 & \text{if } v > 0 \\ m v'(t) &= m \alpha + k v^2 & \text{if } v < 0. \end{aligned}$$

<http://exploration.grc.nasa.gov/education/rocket/termvr.html>

(We'll study the linear and quadratic models in more detail on Friday.)