

Math 2250-4
 Mon Jan 23
 2.1 Improved population models

Final intro to Maple sessions in LCB 115:
 today Monday 9:30-10:20 a.m., 4-4:50 p.m.
 Tuesday 11:35-12:25
 Wednesday 3-3:50 p.m.

Let $P(t)$ be a population at time t . Let's call them "people", although they could be other biological organisms, decaying radioactive elements, accumulating dollars, or even molecules of solute dissolved in a liquid at time t (2.1.23). Consider:

$B(t)$, birth rate (e.g. $\frac{\text{people}}{\text{year}}$);

$\beta(t) := \frac{B(t)}{P(t)}$, fertility rate ($\frac{\text{people}}{\text{year}}$ per person)

$D(t)$, death rate (e.g. $\frac{\text{people}}{\text{year}}$);

$\delta(t) := \frac{D(t)}{P(t)}$, morbidity rate ($\frac{\text{people}}{\text{year}}$ per person)

Then in a closed system (i.e. no migration in or out) we can write the governing DE two equivalent ways:

$$P'(t) = B(t) - D(t)$$

$$P'(t) = (\beta(t) - \delta(t))P(t).$$

Model 1: constant fertility and morbidity rates, $\beta(t) \equiv \beta_0 \geq 0$, $\delta(t) \equiv \delta_0 \geq 0$, constants.

$$\Rightarrow P' = (\beta_0 - \delta_0)P = kP.$$

This is our familiar exponential growth/decay model, depending on whether $k > 0$ or $k < 0$.

Model 2: population fertility and morbidity rates only depend on population P , but they are not constant:

$$\beta = \beta_0 + \beta_1 P$$

$$\delta = \delta_0 + \delta_1 P$$

with $\beta_0, \beta_1, \delta_0, \delta_1$ constants. This implies

$$P' = (\beta - \delta)P = ((\beta_0 - \delta_0) + (\beta_1 - \delta_1)P)P.$$

For viable populations, $\beta_0 > \delta_0$. For a sophisticated (e.g. human) population we might also expect

$\beta_1 < 0$, and resource limitations might imply $\delta_1 > 0$. With these assumptions, and writing $\beta_1 - \delta_1 = -a$

< 0 , $\beta_0 - \delta_0 = b > 0$ one obtains the logistic differential equation:

$$P' = -aP^2 + bP, \text{ or equivalently}$$

$$P' = aP\left(\frac{b}{a} - P\right) = kP(M - P).$$

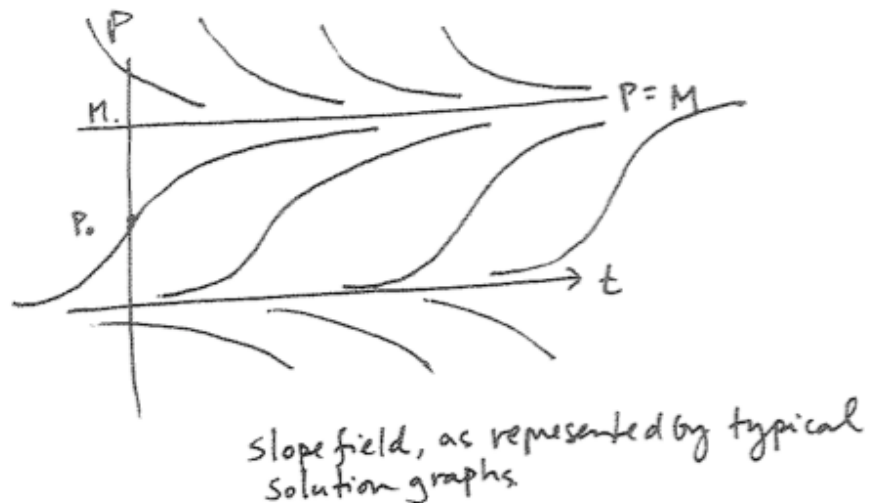
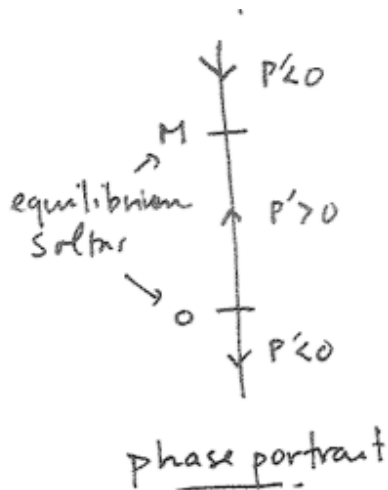
$k = a > 0$, $M = \frac{b}{a} > 0$. (One can consider other cases as well.)

Exercise 1: Discuss qualitative features of the slope field for the logistic differential equation

$$P' = kP(M - P)$$

for $P = P(t)$.

- There are two constant ("equilibrium") solutions. What are they?
- Evaluate the sign and magnitude of the slope function $f(P, t) = kP(M - P)$, in order to understand and be able to recreate the two diagrams below. One is a qualitative picture of the slope field, in the $t - P$ plane. The diagram to the left of it, called the phase diagram, is just a P number line with arrows indicating whether $P(t)$ is increasing or decreasing on the intervals between the constant solutions.
- When discussing the logistic equation, the value M is called the "carrying capacity" of the (ecological or other) system. Discuss why this is a good way to describe M . Hint: if $P(0) = P_0$, and $P(t)$ solves the logistic equation, what is the apparent value of $\lim_{t \rightarrow \infty} P(t)$?



Exercise 2: Solve the logistic DE IVP

$$\begin{aligned}P' &= k P (M - P) \\ P(0) &= P_0\end{aligned}$$

via separation of variables. Verify that the solution formula is consistent with the slope field and phase diagram discussion from exercise 1. Hint: You should find that

$$P(t) = \frac{MP_0}{(M - P_0)e^{-Mkt} + P_0} .$$

Application!

The Belgian demographer P.F. Verhulst introduced the logistic model around 1840, as a tool for studying human population growth. Our text demonstrates its superiority to the simple exponential growth model, and also illustrates why mathematical modelers must always exercise care, by comparing the two models to actual U.S. population data.

```
> restart : # clear memory
  Digits := 5 : #work with 5 significant digits
> pops := [[1800, 5.3], [1810, 7.2], [1820, 9.6], [1830, 12.9],
  [1840, 17.1], [1850, 23.2], [1860, 31.4], [1870, 38.6],
  [1880, 50.2], [1890, 63.0], [1900, 76.2], [1910, 92.2],
  [1920, 106.0], [1930, 123.2], [1940, 132.2], [1950, 151.3],
  [1960, 179.3], [1970, 203.3], [1980, 225.6], [1990, 248.7],
  [2000, 281.4], [2010, 308.]] : #I added 2010 - between 306-313
  # I used shift-enter to enter more than one line of information
  # before executing the command.
> with(plots) : #plotting library of commands
  pointplot(pops, title = `U.S. population through time`);
>
```

Unlike Verhulst, the book uses data from 1800, 1850 and 1900 to get constants in our two models. We let $t=0$ correspond to 1800.

Exponential Model: For the exponential growth model $P(t) = P_0 e^{r \cdot t}$ we use the 1800 and 1900 data to get values for P_0 and r :

```
> P0 := 5.308;
  solve(P0·exp(r·100) = 76.212, r);
                                     P0 := 5.308
                                     0.026643 (1)
> P1 := t→5.308·exp(.02664·t);#exponential model -eqtn (9) page 83
                                     P1 := t→5.308 e0.02664 t (2)
>
```

Logistic Model: We get P_0 from 1800, and use the 1850 and 1900 data to find k and M :

```
> P2 := t→M·P0/(P0 + (M-P0)·exp(-M·k·t)); # logistic solution we worked out
```

$$P2 := t \rightarrow \frac{M P_0}{P_0 + (M - P_0) e^{-M k t}} \quad (3)$$

```
> solve( {P2(50) = 23.192, P2(100) = 76.212}, {M, k});
```

$$\{M = 188.12, k = 0.00016772\} \quad (4)$$

```
> M := 188.12;
k := .16772e-3;
P2(t); #should be our logistic model function,
#equation (11) page 84.
```

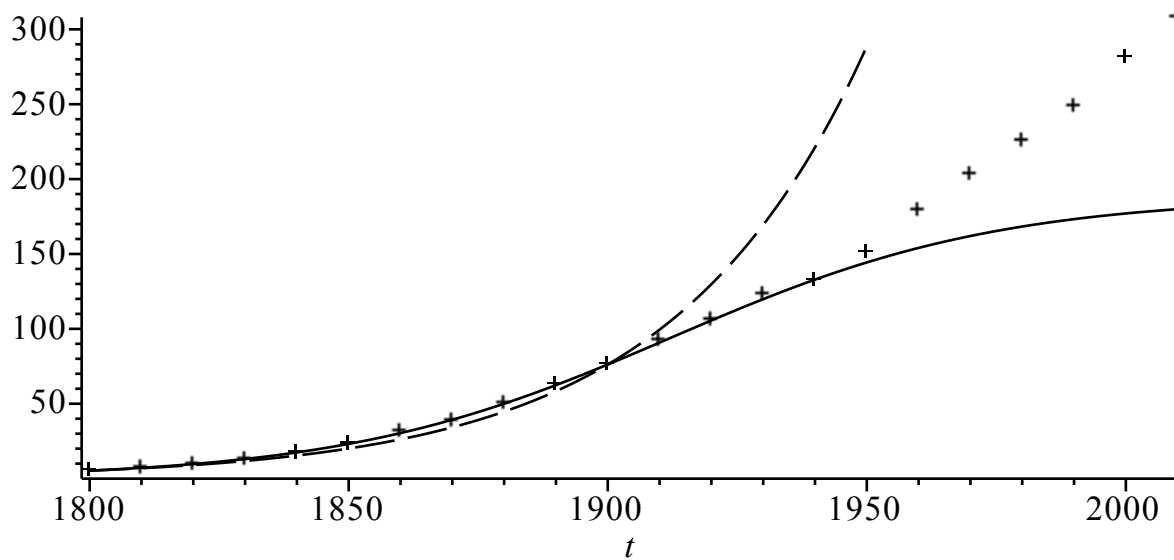
$$\frac{998.54}{5.308 + 182.81 e^{-0.031551 t}} \quad (5)$$

```
>
```

Now compare the two models with the real data, and discuss. The exponential model takes no account of the fact that the U.S. has only finite resources. Any ideas on why the logistic model begins to fail (with our parameters) around 1950?

```
> plot1 := plot(P1(t-1800), t = 1800..1950, color = black, linestyle = 3) :
#this linestyle gives dashes for the exponential curve
plot2 := plot(P2(t-1800), t = 1800..2010, color = black) :
plot3 := pointplot(pops, symbol = cross) :
display( {plot1, plot2, plot3}, title = 'U.S. population data
and models');
```

U.S. population data
and models



```
>
```

