Math 2250-4

Tues Jan 17

1.4: separable DEs, examples and experiment.

1.5: linear DEs intro.

Intro to Maple sessions in LCB 115:

Tues 1/17 (today) 3-3:50 p.m. N. Korevaar

Wed 1/18 11:50-12:40 G.Gustafson, 3-3:50 P. Bardsley

Thur 1/19 5:10-6 p.m. P. Bardsley

Fri 1/20 8:40-9:30 a.m. Kishalaya Saha, 2-3 p.m. TBA

If you choose to buy a copy of Maple for your personal computer, you can use the bookstore or purchase on-line. For an online purchase there is a 25% discount available, which Professor Gustafson has arranged. See

http://www.math.utah.edu/~gustafso/s2012/2250/index2250S2012.html

Exercise 1: To review separable DE's and the existence-uniqueness theorem, understand the solutions to IVP's for the DE:

$$y' = 6 x (y - 1)^{\frac{2}{3}}$$
$$y(x_0) = y_0$$

Are your answers consistent with the slope field and solution trajectories sketched below?
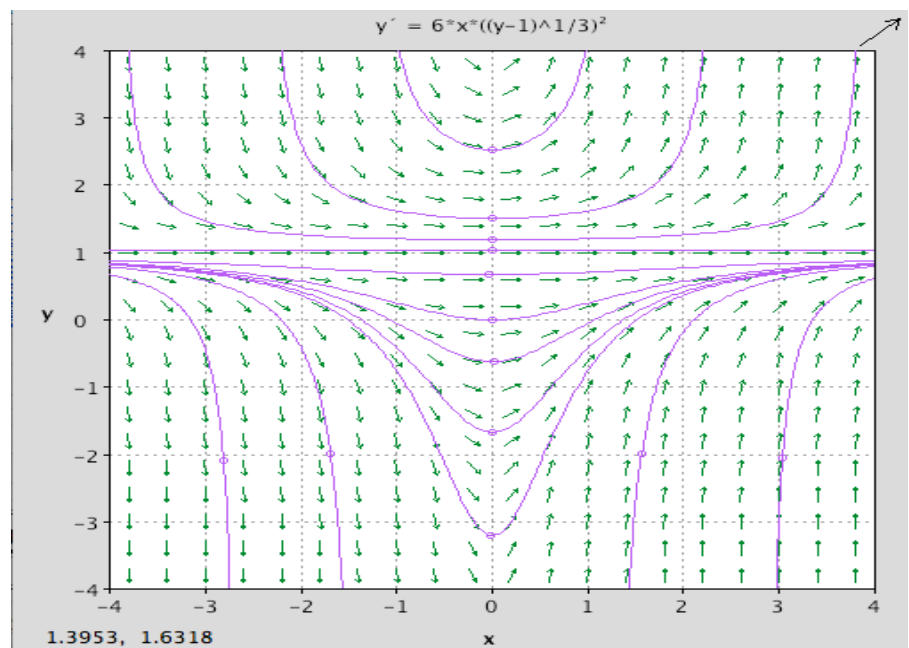
For your hw this week I assigned a selection of separable DE's in section 4 - some applications will be familiar e.g. exponential growth/decay, Newton's Law of cooling, but here's one that might be new to you, and that illustrates important modeling ideas.

Toricelli's Law, for draining water tanks. Refer to the figure below.
Exercise 2:
a) Neglect friction, use conservation of energy, and assume the water still in the tank is moving with negligable velocity $(a << A)$. Equate the lost potential energy from the top in time dt to the gained kinetic energy in the water streaming out of the hole in the tank to deduce that the speed v with which the water exits the tank is given by

$$v = \sqrt{2\,g\,y}$$

when the water depth above the hole is $y(t)$ (and g is accel of gravity).

b) Use part (a) to derive the separable DE for water depth

$$A(y)\,\frac{dy}{dt} = -k\sqrt{y} \quad (k = a\sqrt{2\,g}) .$$

section area $A(y)$

$\frac{1}{7} dy$ (negative)

y

section area a

$v\,dt$
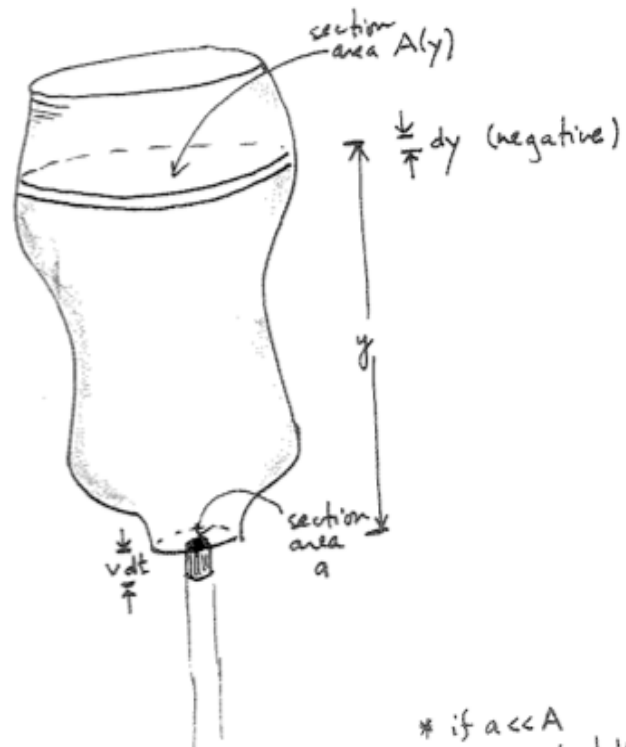
* if $a << A$
we may neglect the
KE of the slow
moving water inside
the vessel.

Experiment fun! I've brought a leaky nalgene canteen so we can test the Toricelli model. For a cylindrical tank of height h as below, the cross-sectional area $A(y)$ is a constant A, so the Toricelli DE and IVP becomes
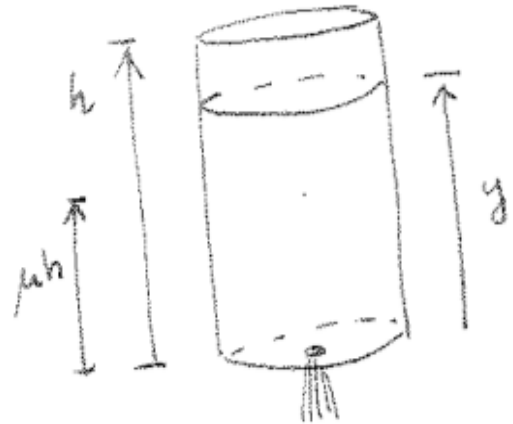
$$\frac{dy}{dt} = -k\, y^{\frac{1}{2}}$$
$$y(0) = h$$

(different k). Let $T(\mu\, h)$ be the time it takes the the water to go from height h (full) to height $\mu \cdot h$, where the fraction μ is between 0 and 1. Note, $T(h) = 0$ and $T(0)$ is the time it takes for the tank to empty completely.

Exercise 3: (We will use this calculation in our experiment) Solve the IVP and then show the time it takes the tank to empty completely is related to $T(\mu\, h)$ by

$$T(0) = \frac{T(\mu\, h)}{1 - \sqrt{\mu}}\ .$$

Experiment! We'll time how long it takes to half-empty the canteen, and predict how long it will take to completely empty it when we rerun the experiment. Here are numbers I once got in my office, let's see how ours compare.

```
> Digits := 5 :
> 1/(1 - sqrt(.5)) ; # the factor from above, when mu is 0.5
> Thalf := 35;   # seconds to half-empty canteen
  Tpredict := 3.4143·Thalf;  #prediction
>
```

Remark: Maple can draw direction fields, although they're not as easy to create as in "dfield". On the other hand, Maple can do any undergraduate mathematics computation, including solving pretty much any differential equation that has a closed form solution. Let's see what the commands below produce. We'll get some error messages related to the existence-uniqueness theorem!

> $with(DEtools)$: # this loads a library of DE commands

> $deqtn1 := y'(t) = -\sqrt{y(t)}$; # took k=1
> $ics1 := y(0) = 1$; # took initial height=1

> $dsolve(\{deqtn1, ics1\}, y(t))$; # DE IVP sol!

> $factor(\%)$; #how we wrote it

> $DEplot(deqtn1, y(t), 0..3, \{[y(0) = 0], [y(0) = 1.], [y(0) = 2.], [y(0) = 0.5], [y(0) = 1.5]\},$
> $arrows = line, color = black, linecolor = black, dirgrid = [30, 30], stepsize = .1, title$
> $= \text{`Toricelli`})$;

>

....

Section 1.5, linear differential equations:

A first order linear DE for $y(x)$ is one that can be written as
$$y' + P(x)y = Q(x)$$

Exercise 4: Which of the two differential equations below are linear? Rewrite the one that is in the form indicated above:

a) $y'(x) = x - y + 1$

b) $y'(x) = x - y^2 + 1$.

Algorithm for solving linear DEs
$$y' + P(x)y = Q(x)$$
Let $\int P(x)dx$ be any antiderivative of P. Multiply both sides of the DE by its exponential to yield an equivalent DE:
$$e^{\int P(x)dx}(y' + P(x)y) = e^{\int P(x)dx}Q(x)$$
because the left side is a derivative (product rule):
$$\frac{d}{dx}\left(e^{\int P(x)dx}y\right) = e^{\int P(x)dx}Q(x).$$
So you can antidifferentiate both sides with respect to x:
$$e^{\int P(x)dx}y = \int e^{\int P(x)dx}Q(x)dx + C.$$

Dividing by the factor $e^{\int P(x)dx}$ yields a formula for $y(x)$.

Exercise 5: Solve the differential equation
$$y'(x) = 2y - 4x,$$
and compare your solutions to the dfield plot below. You may recognize this DE from your first hw assignment (at which time you didn't know how to derive the solutions).

```
> deqtn2 := y'(x) = 2·y(x) - 4·x :
   dsolve(deqtn2, y(x));  # Maple check
```

y´ = 2*y-4*x   Arrow of slope +1.0

-3.9256, 3.7612