

Math 2250-4

Week 7 concepts and homework, due February 24.

Recall that problems which are underlined are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

Because I assume you are proficient in computing reduced row echelon forms, determinants, and inverse matrices by hand, you are free to use technology for the computations in this week's homework, once you set up the problems. With a google search you should be able to find applets that compute reduced row echelon form. Alternately you may use Maple, Matlab, Mathematica, or a graphing calculator.

If you need practice, I'd recommend doing some of these computations by hand and only using technology to check your answers afterwards, however.

4.1: linear combinations of vectors in R^2 and R^3 ; linear dependence and independence; subspaces of R^3 ;
 ;
 1, 7, 9, 10, 15, 16, 22, 25, 26, 33.

w7.1) Consider the three vectors

$$\underline{u} := \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \underline{v} := \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \underline{w} := \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

- Compute the magnitudes (lengths), $|\underline{u}|$ and $|\underline{v}|$.
- Express \underline{w} as a linear combination of \underline{u} and \underline{v} , by solving the appropriate linear system.
- Make a careful and accurate sketch which illustrates your answer to (b), as we did in Exercise 1 of the Friday February 17 class notes. (You can print off free graph paper at <http://www.printfreegraphpaper.com/>)
- Find a linear combination of the three vectors which adds up to the zero vector. (You've already done the work for this in part (c), if you just rearrange your equation!) Illustrate this linear combination adding up to zero on your sketch for (c).

w7.2) Consider the three vectors

$$\underline{u} := \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \quad \underline{v} := \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \quad \underline{w} := \begin{bmatrix} -2 \\ 13 \\ -2 \end{bmatrix}.$$

- Use a reduced row echelon computation to check that these vectors do not span R^3 . For reference in this problem see Exercise 4 in the Friday February 17 notes.
- Use your reduced row echelon form computation to write \underline{w} as a linear combination of $\underline{u}, \underline{v}$.
- The span of these three vectors is actually a plane through the origin. Find the implicit equation $ax + by + cz = 0$ satisfied all points (x, y, z) in the span of $\underline{u}, \underline{v}, \underline{w}$.

w7.3) Consider the three vectors

$$\underline{\mathbf{u}} := \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \quad \underline{\mathbf{v}} := \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \quad \underline{\mathbf{w}} := \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}.$$

a) Compute the reduced row echelon form of the matrix $\langle \underline{\mathbf{u}} | \underline{\mathbf{v}} | \underline{\mathbf{w}} \rangle$ to deduce that every vector $\underline{\mathbf{b}}$ in \mathbb{R}^3 can be expressed uniquely as a linear combination of these three vectors, i.e.

$$c_1 \underline{\mathbf{u}} + c_2 \underline{\mathbf{v}} + c_3 \underline{\mathbf{w}} = \underline{\mathbf{b}}$$

is always uniquely solvable for the linear combination coefficients c_1, c_2, c_3 . (In particular, the only linear combination of the vectors which adds up to $\underline{\mathbf{0}}$ is when $c_1 = c_2 = c_3 = 0$, so the vectors also satisfy the definition of independence.)

b) Compute the determinant of the matrix you worked with in part (a), and explain why the result of (a) also follows from this computation.

4.2: *subspaces of \mathbb{R}^n , expressed either as the span of a collection of vectors and/or as the solution space of a homogeneous matrix equation $A\underline{\mathbf{x}} = \underline{\mathbf{0}}$.*

3, 4, 5, 6, 9, 15, 18, 24, 27, 29;

w7.4) In problem **w7.2** you showed how to go from a spanning set for a plane in \mathbb{R}^3 to its implicit equation $ax + by + cz = 0$. In this problem, you'll do an example of the reverse procedure: Find two linearly independent vectors that span the plane with implicit equation

$$2x - 3y + z = 0.$$

Hint: This single linear equation is a special case of a linear system. If you let two of the variables be free parameters you can solve for the third variable. Write your solution in vector linear combination form and it should be that it's a linear combination of two vectors. Explain why these two vectors are a basis.

w7.5) Consider the somewhat larger homogenous system

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ -1 & -2 & 4 & 2 \\ -1 & -2 & 7 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Express the solution space of this matrix equation (which is a subspace of \mathbb{R}^4) as the span of two vectors.

4.3: testing for independence of vectors; using reduced row echelon form to find dependencies for the columns of the matrix.

1, 3, 6, 9, 10, 16, 17, 18, 23, 25.

w7.6) Consider the four columns of the matrix from problem **w7.5**. Since there are 4 of them and they are vectors in \mathbb{R}^3 you know they must be dependent. But in this case, no three of them are even independent. Using the reduced row echelon form of this matrix, which you have already computed in **w7.5**, write down all possible choices of just 2 of the column vectors which will still span the same plane in \mathbb{R}^3 that all four columns spanned. Hint: use the "magic" fact that column dependencies/independencies do not change when you do elementary row operations, because they correspond to solutions to the homogeneous matrix equation. (We discuss this fact in class, which only seems magic.)

4.4: bases for vector spaces (including subspaces); creating a basis from a spanning set by deleting dependent vectors; adding vectors to an independent set to create a basis; dimension of a vector space.

1, 2, 3, 6, 8, 9, 11, 13, 26.

w7.7)

a) Exhibit a basis for the plane spanned by the three vectors in **w7.2**. What is the dimension of this plane?

b) Find a third vector in \mathbb{R}^3 to augment to the two vectors in your basis from part (a), to create a basis of three vectors for \mathbb{R}^3 . Hint: The standard basis of \mathbb{R}^3 , namely

$\mathbf{e}_1 = [1, 0, 0]^T$, $\mathbf{e}_2 = [0, 1, 0]^T$, $\mathbf{e}_3 = [0, 0, 1]^T$, span \mathbb{R}^3 , so can't all be in the span of the two vectors you start out with. Thus at least one of these will be a suitable choice for your third vector. Augment the two column vectors you started with these three vectors, reduce, and draw conclusions.

c) Exhibit a basis for the solution space to the homogeneous matrix equation in **w7.5**. What is the dimension of this subspace of \mathbb{R}^4 ?

d) Find two more vectors in \mathbb{R}^4 to augment with the two vectors in your basis from part (c), to create a basis for \mathbb{R}^4 .

e) Exhibit a basis for the span of the four columns of the matrix in **w7.5**, made out of two of the original columns. What is the dimension of this subspace of \mathbb{R}^3 ?

w7.8a) In class we'll discuss why doing the three "elementary vector" operations on a set of vectors, namely (i) interchanging their order, (ii) multiplying one of them by a nonzero constant, or (iii) replacing one of them by its sum with a multiple of another one, do not change the span of the set. Use this fact, and start with the original four column vectors of the matrix in **w7.5**, to find a basis for the plane they span which is of the particularly nice form

$$\left\{ \left[\begin{array}{c} 1 \\ 0 \\ * \end{array} \right], \left[\begin{array}{c} 0 \\ 1 \\ * \end{array} \right] \right\}.$$

Hint: This amounts to computing the reduced column echelon form of the matrix in **w7.5**.

b) Express the two vectors you found in the **w7.7e** subspace basis as linear combinations of the two vectors you found in the different basis for the same subspace, in **w7.8a**. You should be able to read off the linear combination coefficients because of the nice form for the basis in **w7.8a**.