Math 2250-4
Week 15 concepts and homework, due April 27.

Recall that problems which are underlined are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

9.1 concepts for autonomous first order systems of DE's: equilibrium solutions, stability, introduction to the names for different types of equilibrium solutions, visualization.
9.1: 5, 8, 11, 15.

9.2 finding equilibrium solutions (aka critical points), and linearizing at the critical point in order to classify it and understand how the non-linear system behaves nearby.
9.2: 5, 7, 9, 13, 15, 17, 19
w15.1) Consider the non-linear system of differential equations from the competition model we discussed in class on April 18 and 20.
$$x'(t) = 14\,x - 2\,x^2 - x\,y$$
$$y'(t) = 16\,y - 2\,y^2 - x\,y\,.$$
We found the four equilibrium (critical point) solutions $(0,0)$, $(7,0)$, $(0,8)$, $(4,6)$ and did a linearization analysis at $(4,6)$, which is a stable node.
a) Linearize the system at the $(7,0)$ using the Jacobian matrix, and classify the critical point according to the eigenvalues of the matrix in the linearized system, using table 9.2.12 on page 539 (or our class notes from April 20).
b) Compute the general solution to the linearized system at $(7,0)$, and use the eigenvalue/eigenvector information to sketch a qualitatively accurate phase portrait in the $u - v$ plane, like we did in class on Wednesday and Friday for the equilibrium points $(4,6)$ and $(0,8)$. (Your picture should look very much like the pplane solution to the non-linear problem near $(u,v) = (0,0)$, i.e. near $(x,y) = (0,8)$.)

9.3: interacting population models: competition and predator-prey modeling
9.3: 5, 7, 11, **18,**19. In 18, the text has a typo in equation (7), it means to read $y' = -5\,y + x\,y$.

w15.2) Use the separation of variables trick in 9.3.19, (i.e. the Calculus fact that $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)}$) to show that solution trajectories follow level curves for a certain function, in order to show that the equilibrium point $(5,2)$ is a stable center for the non-linear autonomous system. (In 19 you verify that it's a stable center for the linearization, but this is a borderline case as far as classifying the equilibrium point stability for the nonlinear problem.)
w15.3) We and the text discuss the fact that for the general logisitic competition model
$$x'(t) = a_1 \cdot x - b_1 \cdot x^2 - c_1 \cdot x\,y$$
$$y'(t) = a_2 \cdot y - b_2 \cdot y^2 - c_2 \cdot x\,y\,,$$
if the product $b_1 b_2$ satisifies $b_1 b_2 > c_1 c_2$ then any first quadrant equilibrium will be asymptotically stable and a global attractor, as in the system for **w15.1** . On the other hand, if $b_1 b_2 < c_1 c_2$ then one of the populations will "always" die out. This is illustrated in the following system:
$$x'(t) = 9\,x - \frac{x^2}{2} - x\,y$$
$$y'(t) = 12\,y - y^2 - x\,y\,.$$

a) Find all equilibrium solutions.

b) Linearize the system about each of the 4 equilibrium solutions, and classify each equilibrium point according to the matrix eigenvalues.

c) For the equilibrium solution $(18, 0)$, find the general solution to the linearized problem, and draw a qualitatively correct (u, v) plane picture using this general solution (use ideas we discusssed on Wed April 18).

d) Use pplane to check your work in this problem: Create a phase portrait with the equilibrium solutions highlighted along representative solution curves. Include the stable and unstable orbits (also known as "seperatrices") from the saddle point in the first quadrant. Print out and hand in a screenshot of this diagram. (You can check your linearization analyses in a,b,c as well, but you don't need to hand this part in.) Explain how the location of the initial population point (x_0, y_0) determines which population dies out and which one survives.

9.4) Nonlinear mechanical systems.

9.4: 9, **10**, 11, 13, **14**

w15.3) Consider the first order system of differential equations

$$x'(t) = y$$
$$y'(t) = -1.25 \sin(x) - y.$$

Notice this first order system arises from the non-linear second order rigid rod pendulum equation

$\theta''(t) + c\,\theta'(t) + \dfrac{g}{L}\sin(\theta(t)) = 0$, with $x(t) = \theta(t)$, in which we have chosen L to make $\dfrac{g}{L} = 1.25$

and added some friction, with $c = 1$. We've discussed (or will discuss, depending on when you read this) the no-damping case in class.

a) What are the equilibrium solutions of the first order system? What swing positions do they correspond to?

b) Linearize this system at $(\pi, 0)$. Compute the general solution to the linearized problem, classify the critical point, and sketch a qualititatively accurate picture of the phase portrait for the linearized problem. Recreate the phase portrait for this system on a suitable scale, create representative solution trajectories, and plot the stable and unstable orbits from $(\pi, 0)$. At which two possible equilibrium points will solutions that start near to $(\pi, 0)$ end up, with 100 % probability, based on the phase portrait for the non-linear problem? What is the interpretation of this in terms of the rigid-rod swing?

c) Linearize the system at $(0, 0)$, classify the critical point, and sketch a qualititatively correct picture of the linearized problem near there. (As we discuss in class, you need not find the actual eigenvectors - you can just see what the tangent field looks like near $(0, 0)$ to deduce the direction the spiral is rotating.)

d) Indicate on your pplane plot the region of initial values in the phase plane for which solutions ultimately converge to $(0, 0)$.

w15.4) This is a continuation of **14**. Use the $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)}$ trick and separation of variables trick to get the (multiple of) total energy function for which the solution trajectories are following the level curves. Use software to graph the total energy function, and verify visually that the level curves do correspond to the solution trajectories as shown in figure 9.4.12. Hand in a picture of this graph $z = f(x, y)$.