Name..

I.D. number...

Math 2250-4
FINAL EXAM SOLUTIONS
April 3, 2012

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. Laplace Transform tables are included with this exam. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 150 points possible, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

problem	score	possible
1	_______	20
2	_______	20
3	_______	15
4	_______	15
5	_______	10
6	_______	10
7	_______	15
8	_______	15
9	_______	30
total	_______	150

1) Suppose that an object moves vertically, subject only to the acceleration of gravity $g = 32\ \frac{ft}{s^2}$ and a drag force proportional to the object's velocity. Choose the positive y direction to be up and write $y'(t) = v(t)$ for the velocity. For particular values of the object's mass and the drag coefficient, the differential equation

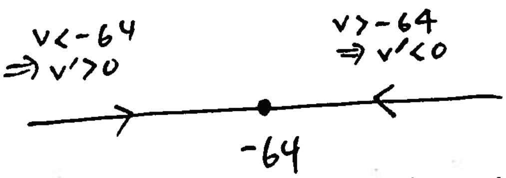
$$\frac{dv}{dt} = -32 - .5\,v$$

governs the object's velocity $v(t)$.

1a) Construct a phase diagram and determine $\lim_{t \to \infty} v(t)$ for all solutions to this differential equation. What is the term for this limiting velocity?

(5 points)

$\frac{dv}{dt} = -.5\,(v + 64)$



So $\lim_{t \to \infty} v(t) = -64$ terminal velocity

1b) Suppose an object is thrown vertically upwards so that its velocity satisfies

$$\frac{dv}{dt} = -32 - .5\,v$$
$$v(0) = 20\,.$$

Find a formula for $v(t)$.

(10 points)

$\frac{dv}{dt} + .5\,v = -32$

$e^{.5t}(v' + .5v) = e^{.5t}(-32)$

$e^{.5t}v = \int -32e^{.5t}\,dt = -64e^{.5t} + C$

$\div e^{.5t}$ $\quad v(t) = -64 + Ce^{-.5t}$

$v(0) = 20 \Rightarrow C = 84$

$\boxed{v(t) = -64 + 84e^{-.5t}}$

1c) Assume the height $y(t)$ of this thrown object satisfies $y(0) = 0$. Find its the maximum height.

(5 points)

$y(t) = y_0 + \int_0^t v(s)\,ds$

$= 0 + \int_0^t -64 + 84e^{-.5s}\,ds$

$= -64s - 168e^{-.5s}\Big]_0^t = -64t + 168(1 - e^{-.5t}).$

y_{max} when $v(t) = 0$

i.e. $64 = 84e^{-.5t}$

$\frac{16}{21} = \frac{64}{84} = e^{-.5t}$

$\boxed{-2\ln\left(\frac{16}{21}\right) = t}$ plug this t value into $y(t)$ to get y_{max}.

2) Consider the following initial value problem, which could arise from an unforced mass-spring oscillation problem:

$$x''(t) + 2x'(t) + 10x(t) = 0$$
$$x(0) = 2$$
$$x'(0) = 4\,.$$

2a) What sort of damping is exhibited in this problem?

(2 points)

$$p(r) = r^2 + 2r + 10 = (r+1)^2 + 9 = 0$$
$$(r+1)^2 = -9$$
$$r+1 = \pm 3i$$
$$r = -1 \pm 3i$$

underdamped

2b) Solve this initial problem using the methods of Chapter 5, i.e. by using the characteristic polynomial method.

(13 points)

from $p(r)$ roots, deduce

$$x(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t$$
$$x(0) = 2 = c_1$$
$$x'(0) = -c_1 + 3c_2 = 4 \Rightarrow -2 + 3c_2 = 4 \Rightarrow 3c_2 = 6 \Rightarrow c_2 = 2$$

$$\boxed{x(t) = 2e^{-t}\cos 3t + 2e^{-t}\sin 3t}$$

2c) Explain what your solution to 2b) has to do with the phase portrait and curve shown below. Your explanation should include an explanation of what the system of first order differential equations shown in the pplane output has to do with the second order differential equation in this problem.

(5 points)

let $x = x(t)$
$y = x'(t)$

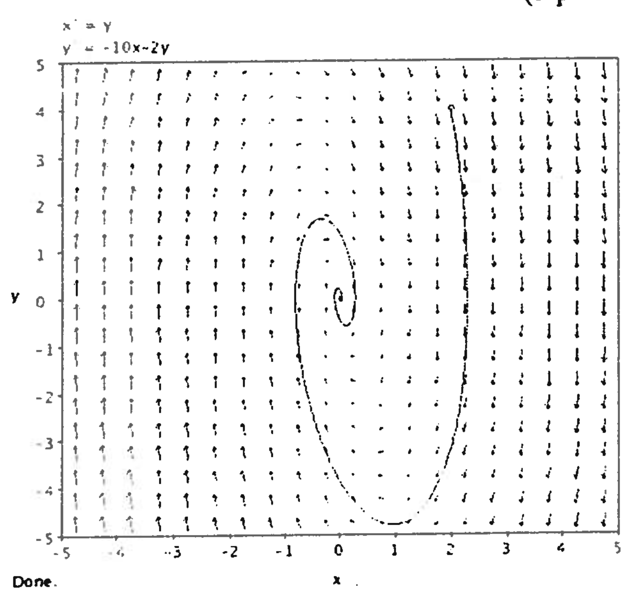
then if $x(t)$ solves
$x'' + 2x' + 10x = 0$

we deduce

$$x' = y$$
$$y' = x'' = -2x' - 10x$$
$$= -10x - 2y$$

So $\begin{bmatrix} x \\ x' \end{bmatrix}$ solves $\begin{aligned} x' &= y \\ y' &= -10 - 2y. \end{aligned}$

If $x(0)=2$, $x'(0)=4$
then $x(0)=2$
$y(0)=4$



So the parametric curve is the output of $\begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}$ for $t \geq 0$. (Since it begins at $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$).

3a) Find the general solution to the forced mass-spring problem

$$x''(t) + 2x'(t) + 10x(t) = 5 + 10t.$$

Notice that you already found the homogeneous solution in problem 2).

(10 points)

for x_p try $x_p = A + Bt$

$x_p' = B$

$x_p'' = 0$

So $x_p'' + 2x_p' + 10x_p = 0 + 2B + 10(A+Bt) = 5 + 10t$

$(2B + 10A) + (10B)t = 5 + 10t$

$\Rightarrow 2B + 10A = 5$

$10B = 10$

$\Rightarrow B = 1 \Rightarrow 2 + 10A = 5 \Rightarrow A = \frac{3}{10}.$

$\Rightarrow x = x_p + x_H = \frac{3}{10} + t + c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t$

3b) Use Laplace transform to solve a different forcing problem for this configuration. In this problem the mass is at equilibrium until time $t = 2$. At this time it is forced with an impulsive force such that $x(t)$ satisfies

$$x''(t) + 2x'(t) + 10x(t) = 3\,\delta(t-2).$$
$$x(0) = 0$$
$$x'(0) = 0.$$

(5 points)

$s^2 X(s) + 2sX(s) + 10X(s) = 3e^{-2s}$

$X(s)(s^2 + 2s + 10) = 3e^{-2s}$

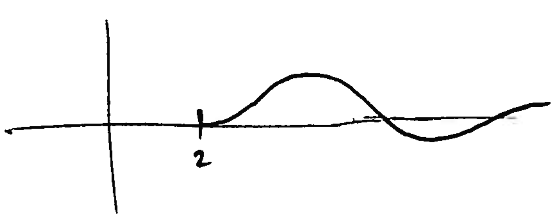
$X(s) = \frac{3}{(s+1)^2 + 9} e^{-2s}$

$u(t-a)f(t-a)$	$e^{-as}F(s)$
$e^{-at}\sin kt$	$\frac{k}{(s-a)^2 + k^2}$

for $F(s) = \frac{3}{(s+1)^2 + 9}$

$f(t) = e^{-t}\sin 3t$

$\Rightarrow x(t) = u(t-2)\, e^{-(t-2)} \sin 3(t-2)$ i.e.



$$= \begin{cases} 0 & 0 \le t \le 2. \\ e^{-(t-2)}\sin 3(t-2) & t \geq 2 \end{cases}$$

4) Consider the forced oscillator initial value problem

$$x''(t) + \omega_0^2 x(t) = F_0 \cos(\omega t).$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

Assume $\omega \neq \omega_0$, i.e. the non-resonance case. Use Laplace transforms to solve the initial value problem.

(15 points)

$$s^2 X(s) - s x_0 - v_0 + \omega_0^2 X(s) = F_0 \frac{s}{s^2+\omega^2}$$

$$X(s)\left[s^2+\omega_0^2\right] = F_0 \frac{s}{s^2+\omega^2} + s x_0 + v_0$$

$$X(s) = F_0 s \frac{1}{(s^2+\omega^2)(s^2+\omega_0^2)} + \frac{s x_0}{s^2+\omega_0^2} + \frac{v_0}{s^2+\omega_0^2}$$

$$= \underbrace{F_0 s\left[\frac{1}{\omega_0^2-\omega^2}\left(\frac{1}{s^2+\omega^2} - \frac{1}{s^2+\omega_0^2}\right)\right]} + \frac{s x_0}{s^2+\omega_0^2} + \frac{v_0}{\omega_0}\frac{\omega_0}{s^2+\omega_0^2}$$

$$= \frac{F_0}{\omega_0^2-\omega^2}\left[\frac{s}{s^2+\omega^2} - \frac{s}{s^2+\omega_0^2}\right]$$

$$\Rightarrow \boxed{x(t) = \frac{F_0}{\omega_0^2-\omega^2}\left[\cos\omega t - \cos\omega_0 t\right] + x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

5) Find a basis for $\mathbb{R}^2$ consisting of eigenvectors for the matrix

$$A = \begin{bmatrix} -4 & 2 \\ 3 & -3 \end{bmatrix}.$$

Make sure to check your answers, because you'll be using them in following problems. Hint: the eigenvalues are integers.

(10 points)

$$|A-\lambda I| = \begin{vmatrix} -4-\lambda & 2 \\ 3 & -3-\lambda \end{vmatrix} = (\lambda+4)(\lambda+3)-6$$

$$= \lambda^2+7\lambda+12-6$$

$$= \lambda^2+\cancel{x}7\lambda+6 = (\lambda+1)(\lambda+6)$$

$\lambda=-1$

$$\left[\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

$\lambda=-6$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

so, $\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are a basis for $\mathbb{R}^2$ consisting of eigenvectors of A.

check: $\begin{bmatrix} -4 & 2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} = -1\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ ✓

$\begin{bmatrix} -4 & 2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \end{bmatrix} = -6\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ✓

6a) Use your work from problem 5) to find the general solution to this first order system of differential equations:

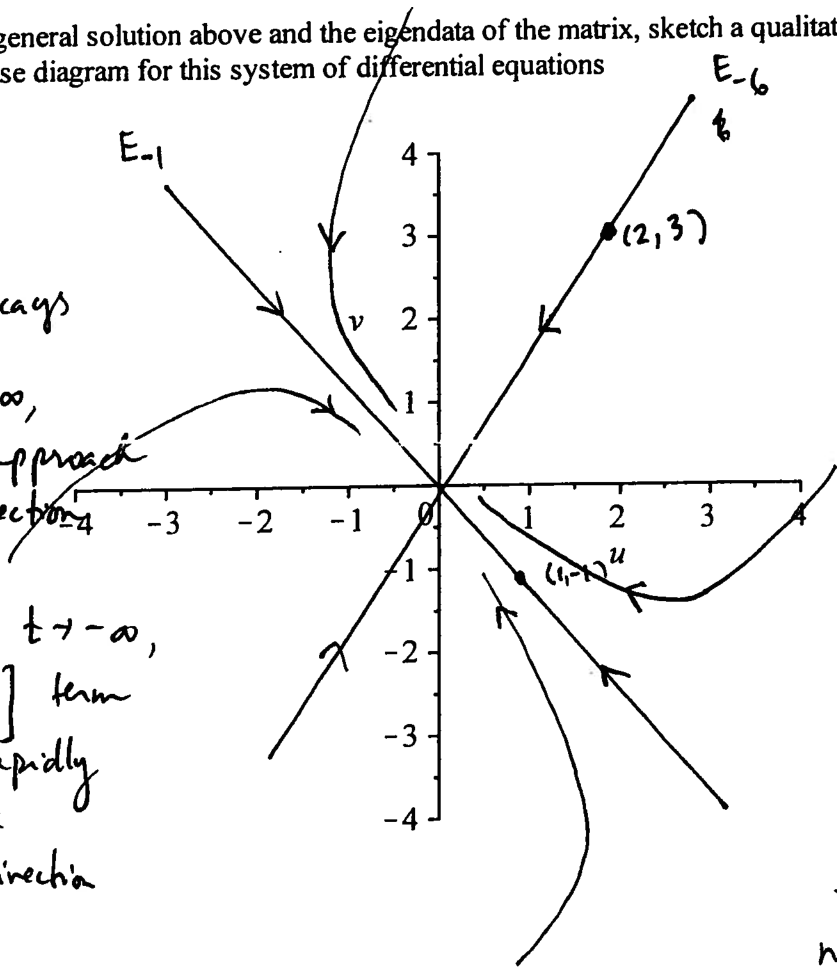
$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}.$$

(5 points)

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

6b) Using your general solution above and the eigendata of the matrix, sketch a qualitatively accurate picture of the phase diagram for this system of differential equations

(5 points)

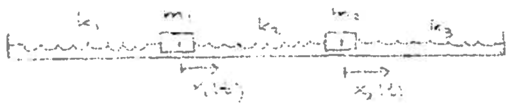


if $c_1, c_2 \neq 0$ then e^{-6t} decays quicker than e^{-t} as $t \to \infty$, so soltn's approach origin from direction of E_{-1}.

conversely, as $t \to -\infty$, the $c_2 e^{-6t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ term grows most rapidly so they come from E_{-6} direction

this is a nodal ~~source~~ sink, (at the origin).

7) Consider the following undamped mass-spring configuration, where as usual $x_1(t), x_2(t)$ measure the displacement of the two masses from equilibrium.



7a) In case $m_1 = 3, m_2 = 2, k_1 = k_2 = 6, k_3 = 0$ (in other words, the third spring isn't really there), show that the resulting system of differential equations reduces to

$$x_1''(t) = -4x_1 + 2x_2$$
$$x_2''(t) = 3x_1 - 3x_2.$$

(4 points)

$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1) \Rightarrow 3x_1'' = -6x_1 + 6(x_2 - x_1) = -12x_1 + 6x_2$

$\Rightarrow x_1'' = -4x_1 + 2x_2$ ✓

$m_2 x_2'' = -k_2(x_2 - x_1) - k_3 x_2 \Rightarrow 2x_2'' = -6x_2 + 6x_1 \Rightarrow x_2'' = 3x_1 - 3x_2$ ✓

7b) What is the dimension of the solution space to this system of DEs? Explain.

(4 points)

4; it's equivalent to a first order system of 4 DE's.
alternately, IVP has unique solutions for 4 initial conds $x_1(0)$, $x_1'(0)$, $x_2(0)$, $x_2'(0)$

7c) Find the general solution to this system of differential equations. Hint: Use the results of problem 5).

(4 points)

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\lambda_1 = -1 \Rightarrow \omega_1 = 1$, $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\lambda = -6 \Rightarrow \omega = \sqrt{6}$, $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$\Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = (c_1 \cos t + c_2 \sin t) \begin{bmatrix} 2 \\ 3 \end{bmatrix} + (c_3 \cos\sqrt{6}\,t + c_4 \sin\sqrt{6}\,t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

7d) Describe the two fundamental modes of this system.

(3 points)

the slow mode, $\omega = 1$, has both masses oscillating in phase, with amplitude ratio 2:3 for x_1 to x_2 amplitudes.

the fast mode, $\omega = \sqrt{6}$ has masses oscillating out of phase, with equal amplitudes

8) Although we usually use a mass-spring configuration to give context for studying second order differential equations, we've also used the rigid-rod pendulum to effectively exhibit several key ideas from this course. Recall that in the undamped version of this configuration, we let the pendulum rod length be L, assume the rod is massless, and that there is a mass m attached at the end of the pendulum, on which the vertical gravitational force acts with force magnitude mg. This mass will swing along circular arcs of signed arclength $s = L\,\theta$ from the vertical reference point where θ is the angle in radians from vertical. The configuration is indicated below. Note that we are considering a pendulum for which the mass (and rod) are able to rotate freely about the fixed end of the rod.

8a) The undamped pendulum system satisfies conservation of energy, i.e. the sum of the kinetic energy from the mass motion plus the potential energy from its change in height must be constant once the system is put into motion. Use this fact to derive the differential equation for $\theta(t)$ that we've studied in this course, namely $\theta''(t) + \frac{g}{L}\sin\theta = 0$

Hints: Express the sum of the kinetic energy and potential energy in terms of $m, L, g, \theta(t), \theta'(t)$. Then use the fact that the total energy is constant in time (conservation of energy) if and only if the time derivative of the total energy is identically zero.

(8 points)

$s = L\theta(t)$

$\Rightarrow v = s'(t) = L\theta'(t)$

height from bottom $= L(1-\cos\theta)$

$\Rightarrow TE = KE + PE = \frac{1}{2}mv^2 + mgh$

$= \frac{1}{2}m\,(L\theta'(t))^2 + mgL\,(1-\cos\theta(t))$

$0 = \frac{d}{dt}TE = mL^2\theta'\theta'' + mgL(\sin\theta\,\theta')$

$0 = mL\theta'[L\theta'' + g\sin\theta] \Rightarrow L\theta'' + g\sin\theta = 0 \Rightarrow \theta'' + \frac{g}{L}\sin\theta = 0$

8b) How did we justify replacing this non-linear differential equation with a linear one when $\theta(t)$ was near the equilibrium solution $\theta \equiv 0$, and what is the linear differential equation we came up with?

(4 points)

$\sin\theta \approx \theta$ for θ small, since $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$

So the linearized DE is $\theta'' + \frac{g}{L}\theta = 0.$

8c) Use the linear differential equation in part 8b) to find the expected period for an oscillating pendulum in the case of small oscillations, in terms of g, L.

(3 points)

$\theta'' + \omega_0^2\theta = 0$ has sol'ns $C\cos(\omega_0 t - \alpha)$

So $\omega_0 = \sqrt{\frac{g}{L}}$ and $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{L}{g}}$

9) Consider this autonomous system of differential equations for two interacting populations $x(t), y(t)$:

$$x'(t) = x - 2x^2 + xy$$
$$y'(t) = y - y^2 + xy$$

9a) Explain why someone would be justified in calling this a "symbiotic logistic populations" model. Your explanation should consider the various terms in these two differential equations, and what they represent. (The word "symbiotic" means that the presence of each species is beneficial to the other species.)

(4 points)

- Since logistic DE is of form $x' = ax - bx^2$ with $a, b > 0$ each population grows logistically without the other.

- the coefficients of the interaction terms ("xy" in each case) are positive in each eqtn, so the population growth rates of each population improve if the other population is present

9b) Find the four equilibrium solutions algebraically. Plot the equilibrium points onto the phase portrait below.

(6 points)

$$x - 2x^2 + xy = 0 \Rightarrow x(1 - 2x + y) = 0$$
$$y - y^2 + xy = 0 \Rightarrow y(1 - y + x) = 0.$$

$x = 0$

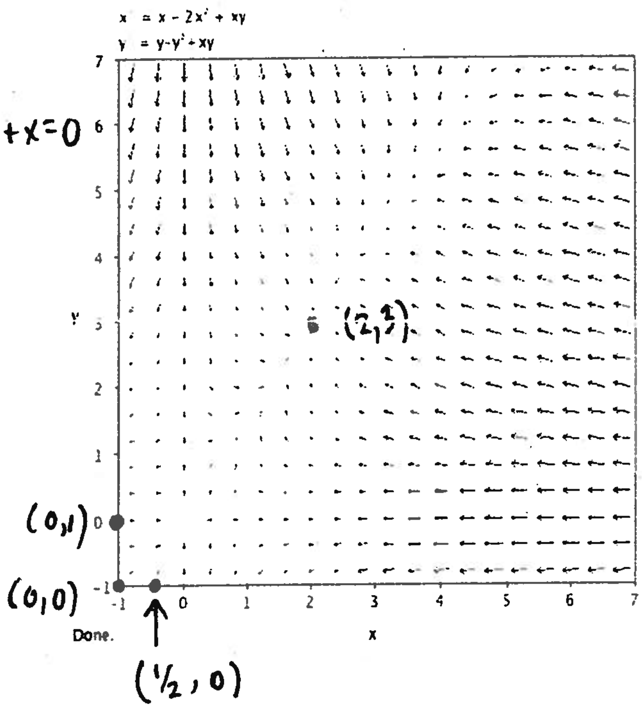
& $y = 0 \Rightarrow (0,0)$

& $1 - y + x = 0 \Rightarrow y = 1 \Rightarrow (0,1)$

$1 - 2x - y = 0$

& $y = 0 \Downarrow x = \frac{1}{2}$ $(\frac{1}{2}, 0)$

& $1 - y + x = 0$



$$1 - 2x + y = 0$$
$$1 - y + x = 0$$

$$2x - y = 1$$
$$x - y = -1$$

$E_1 - E_2 \Rightarrow x = 2 \Rightarrow y = 3$

$(2,3)$

System of DEs repeated for convenience:

$$x'(t) = x - 2x^2 + xy \quad = F$$
$$y'(t) = y - y^2 + xy \quad = G$$

9c) Use the Jacobian matrix to classify each of the equilibrium points. Recall that your choices are: spiral source, spiral sink, stable center, nodal source, nodal sink, saddle point. Your description should include whether or not the equilibrium point is stable. Hints: If you do the algebra correctly and you have the correct equilibrium points none of the eigenvalues are messy. Also, your answers should be consistent with the phase portrait on the previous page.

(16 points)

$(0,0)$, $(0,1)$, $(\tfrac{1}{2},0)$, $(2,3)$

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 1-4x+y & x \\ y & 1-2y+x \end{bmatrix}.$$

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = 1, \left(\text{e.g. } \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \Rightarrow$$ ~~asympt. stable~~ unstable nodal source

$$J(\tfrac{1}{2},0) = \begin{bmatrix} -1 & \tfrac{1}{2} \\ 0 & \tfrac{3}{2} \end{bmatrix} \Rightarrow \lambda_1 = -1,\ \lambda_2 = \tfrac{3}{2}$$ unstable saddle

$$J(0,1) = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \Rightarrow \lambda = -1, 2 \Rightarrow$$ unstable saddle

(did in #6)

$$J(2,3) = \begin{bmatrix} 1-8+3 & 2 \\ 3 & 1-6+2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 3 & -3 \end{bmatrix}. \quad \begin{vmatrix} -4-\lambda & 2 \\ 3 & -3-\lambda \end{vmatrix} = (\lambda+4)(\lambda+3) - 6 = \lambda^2 + 7\lambda + 6 = (\lambda+6)(\lambda+1)$$

$\lambda = -1, -6$ asymp. stable nodal sink

9d) If your equilibrium point and Jacobian computations above are correct, then you will see that the linearized system of differential equations at one of the equilibrium points is the same system you studied in problem 6). Explain how its solution and the phase portrait you drew in 6) are related to the solutions of the non-linear system, and to the phase portrait in the non-linear system.

(4 points)

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

See #6! Since we linearized by setting

$x = 2 + u$
$y = 3 + v$,

$(u,v) = (0,0)$ corresponds to $(x,y) = (2,3)$.

If we magnify the phase diagram on previous page at $(2,3)$ it will look more and more like the one for the linearized problem in #6.