Math 2250-4
Tues Feb 7
3.1-3.3, with hints of matrix algebra 3.4

• Finish exercises 4-6 on Monday's notes. The "summing up" questions of exercise 6 are important enough to repeat them in today's notes as

Exercise 1: **What are your general conclusions?**
1a) What conditions on the reduced row echelon form of the matrix A guarantee that the homogeneous equation $A\underline{x} = \underline{0}$ has infinitely many solutions?
1b) What conditions on the dimensions of A (i.e. number of rows and number of columns) always force infinitely many solutions to the homogeneous problem?
1c) What conditions on the reduced row echelon form of A guarantee that solutions $\underline{x}$ to $A\underline{x} = \underline{b}$ are always unique (if they exist)?

1d) If A is a square matrix ($m=n$), what can you say about the solution set to $A\underline{x} = \underline{b}$ when
  ∗ The reduced row echelon form of A is the identity matrix?
  ∗ The reduced row echelon form of A is not the identity matrix?

Here's a "typical" reduced row echelon form shape (for the unaugmented matrix A) to refer to during our discussion:

$$\begin{bmatrix} 1 & * & 0 & 0 & * & 0 \\ 0 & 0 & 1 & 0 & * & 0 \\ 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Applications of linear systems theory:

Exercise 2: Consider the 2nd order DE for $y = y(x)$:
$$y'' + 4 y' + 3 y = 6 .$$

2a) Show $y(x) = 2 + c_1 e^{-x} + c_2 e^{-3 x}$ solves this DE, no matter what values the free parameters c_1, c_2 have.

2b) Find values of c_1, c_2 to solve the initial value problem for this DE, with initial values
$$y(0) = 4$$
$$y'(0) = 1 .$$

2c) What if the initial values were
$$y(0) = y_0$$
$$y'(0) = v_0 ?$$

<u>Exercise 3:</u> Consider the following combustion equation. Methane, CH_4, is the primary component of natural gas. We've all heard about the dangers of carbon monoxide (CO) buildup if natural gas is burned without enough oxygen O_2 , and that's related to:

$$a \ CH_4 + b \ O_2 \rightarrow c \ CO_2 + d \ H_2O + e \ CO.$$

<u>3a)</u> Balance the equation to deduce that

$$a = c + e$$
$$4 \ a = 2 \ d$$
$$2 \ b = 2 \ c + d + e \ .$$

<u>3b)</u> Order the variables c, d, e, a, b and write this homogeneous linear system in synthetic form. (Since there are 5 variables and only 3 equations, I expect that I'll be able to solve for the outputs c, d, e in terms of the inputs a, b .) Compute the reduced row echelon form and solve for c, d, e in terms of a, b.
<u>3e)</u> Discuss the implications of your work.

$$a \ CH_4 + b \ O_2 \rightarrow (\text{-}3 \ a + 2 \ b) \ CO_2 + 2 \ a \ H_2O + (4 \ a - 2 \ b) \ CO$$

Matrix vector algebra, so far:

Vector addition and scalar multiplication:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ \vdots \\ x_n + y_n \end{bmatrix} ; \qquad c \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} c\,x_1 \\ c\,x_2 \\ c\,x_3 \\ \vdots \\ c\,x_n \end{bmatrix}$$

Vector dot product , which yields a scalar (i.e. number) output (regardless of whether vectors are column vectors or row vectors):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} := x_1 y_1 + x_2 y_2 + \ldots + x_n y_n.$$

Matrix times vector: If A is an $m \times n$ matrix and $\underline{x}$ is an n column vector, then

$$A\underline{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} Row_1(A) \cdot \underline{x} \\ Row_2(A) \cdot \underline{x} \\ \vdots \\ Row_m(A) \cdot \underline{x} \end{bmatrix}$$

Compact way to write our usual linear system:

$$A\underline{x} = \underline{b}.$$

Exercise 4: Check that matrix multiplication distributes over vector addition and scalar multiplication, i.e.

$$A(\underline{x} + \underline{y}) = A\underline{x} + A\underline{y}$$
$$A(c\,\underline{x}) = c\,A\underline{x}$$

<u>Exercise 5</u> (relates to hw): Consider the matrix equation
$$A\underline{x} = \underline{b}.$$
Assume the system is consistent, so that there is at least one (particular) solution $\underline{x}_P$, i.e.
$$A\underline{x}_P = \underline{b}.$$
Use the algebra properties of the previous page to show that

<u>5a)</u> If $\underline{x}_H$ solves the homogeneous system $A\underline{x} = \underline{0}$, then $\underline{x} = \underline{x}_P + \underline{x}_H$ solves the original problem
$$A\underline{x} = \underline{b}.$$
<u>5b)</u> Show that *every* solution to $A\underline{x} = \underline{b}$ is the sum of the given $\underline{x}_P$ with *some* solution to the homogeneous system.

<u>Remark:</u> This answers the question from Exercise 2 on yesterday's (Monday's) notes, where we noticed that the general solution to the inhomogeneous system was a particular solution plus the general solution to the homogeneous system.

(The beauty of abstraction...this may seem strange now, but by the end of the course it will make complete sense.)

Operators that take sums to sums, and scalar multiples to scalar multiples are called "linear". For example, matrix multiplication is linear. But differential operators can also be linear. For example, if we consider our friend the (left-hand side of the) first order linear differential equation for $y = y(x)$,
$$L(y) := y' + p\,y$$
where $p(x)$ is a function, the L is a linear operator:
$$L(y_1 + y_2) := (y_1 + y_2)' + p(y_1 + y_2) = y_1' + y_2' + p\,y_1 + p\,y_2$$
$$= (y_1' + p\,y_1) + (y_2' + p\,y_2) = L(y_1) + L(y_2).$$
$$L(c\,y_1) = (c\,y_1)' + p(c\,y_1) = c\,y_1' + c\,p\,y_1 = c\,L(y_1).$$
Thus the same reasoning applies, as in the matrix system discussion of the previous page:

- The general solution to the first order DE
$$L(y) := y' + p\,y = q$$
is of the form $y = y_P + y_H$ where y_P is any particular solution and y_H is the general solution to the homogeneous DE
$$y' + p\,y = 0 \,.$$

Exercise 6: Use the reasoning above to write down the general solution $y = y(x)$ to our favorite linear differential equation
$$y' + a\,y = b$$
where a, b are constants.
(If you look back at page 2 you'll see that something similar was going on there too.)