

### 3.3 Reduced row echelon form, and consequences for the solution sets of linear systems of equations.

On Friday we chose a fairly large system of linear equations in order to practice the algorithm of computing the reduced row echelon form of the augmented matrix. The system we began with is

$$\begin{aligned}x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 &= 7 \\3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 &= 27.\end{aligned}$$

The corresponding augmented matrix is

$$\left[ \begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array} \right]$$

We worked from the top row down, and from the left column to the right, using elementary row operations to put the augmented matrix into

row-echelon form:

- (1) All "zero" rows (having all entries = 0) lie beneath the non-zero rows.
- (2) The leading (first) non-zero entry in each non-zero row lies strictly to the right of the one above it:

$$\left[ \begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array} \right]$$

(At this stage you could "backsolve" to find all solutions.)

Then we continued with Gaussian elimination, but working from bottom to top and from right to left so that we ended with an augmented matrix that was in

reduced row echelon form: (1),(2), together with

- (3) Each leading non-zero row entry has value 1. Such entries are called "leading 1's"
- (4) Each column that has (a row's) leading 1 has 0's in all the other entries.

$$\left[ \begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array} \right]$$

Exercise 1: Explicitly specify the solution set to the original system of equations by backsolving the reduced row echelon form of the augmented matrix. (You may wish to check that you can get the same solutions from the row echelon form, but that it's a bit messier to work out.)

Solution: After you write your solution in vector form and use vector addition and scalar multiplication to rewrite it, you might get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 7 \\ 0 \end{bmatrix} + q \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -2 \\ 4 \\ 1 \end{bmatrix}; \quad q, t \in \mathbb{R}.$$

Note: There are lots of row-echelon forms for a matrix, but only one reduced row-echelon form. All mathematical software will have a command to find the reduced row echelon form. For example, in Maple:

```
> with(LinearAlgebra) : # matrix and linear algebra library
> A := Matrix(3, 5, [1, -2, 3, 2, 1,
                    2, -4, 8, 3, 10,
                    3, -6, 10, 6, 5]) :
b := Vector([10, 7, 27]) :
<A|b>; # the mathematical augmented matrix doesn't actually have
      # a vertical line between the end of A and the start of b
ReducedRowEchelonForm(<A|b>); #command is long to write, but
      # easy to remember!
```

$$\begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{bmatrix}$$

(1)

```
> LinearSolve(A, b);
# this command will actually write down the general solution, using
# Maple's way of writing free parameters, which actually makes
# some sense. Generally when there are free parameters involved,
# there will be equivalent ways to express the solution that may
# look different. But usually Maple's version will look like yours.
```

Exercise 2) Coefficient matrix taken from problem #19, section 3.3, page 174. (You are assigned #20 to work by hand and check with technology).

```
> A := Matrix(3, 5, [2, 7, -10, -19, 13,
                    1, 3, -4, -8, 6,
                    1, 0, 2, 1, 3]);
```

$$A := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 \\ 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \end{bmatrix} \quad (2)$$

```
> ReducedRowEchelonForm(A);
```

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (3)$$

Let's consider three different linear systems for which A is the coefficient matrix. In the first one, the right hand sides are all zero (what we call the "homogeneous" problem), and I have carefully picked the other two right hand sides. The three right hand sides are separated by the dividing line below:

$$\left[ \begin{array}{ccccc|ccc} 2 & 7 & -10 & -19 & 13 & 0 & 7 & 7 \\ 1 & 3 & -4 & -8 & 6 & 0 & 0 & 3 \\ 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \end{array} \right].$$

We'll try solving three linear systems at once!

```
> b1 := Vector([0, 0, 0]) :
   b2 := Vector([7, 0, 0]) :
   b3 := Vector([7, 3, 0]) :
   C := <A|b1|b2|b3>; # very augmented matrix
   ReducedRowEchelonForm(C);
```

$$C := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 & 0 & 7 & 7 \\ 1 & 3 & -4 & -8 & 6 & 0 & 0 & 3 \\ 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (4)$$

2a) Find the solution sets for each of the three systems:

**Important conceptual questions:**

2b) Which of these three solutions could you have written down just from the reduced row echelon form of  $A$ , i.e. without using the augmented matrix? Why?

2c) Linear systems in which right hand side vectors equal zero are called homogeneous linear systems. Otherwise they are called inhomogeneous or nonhomogeneous. Notice that the general solution to the consistent inhomogeneous system is the sum of a particular solution to it, together with the general solution to the homogeneous system!!! Was this an accident? You'll be thinking about this question in your homework for this week. It's related to an important general concept which will keep coming up in the rest of the course.

Exercise 3) The reduced row echelon form of a (non-augmented) matrix  $A$  can tell us a lot about the solution set to the linear system  $A\mathbf{x}=\mathbf{b}$ . (Do you remember this "matrix times vector" notation for writing a linear system?) Consider the matrix  $A$  below, and answer all questions:

$\text{> } A := \text{Matrix}(2, 5, [2, 7, -10, -19, 13, 1, 3, -4, -8, 6]);$   
 $\text{ReducedRowEchelonForm}(A);$

$$A := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 \\ 1 & 3 & -4 & -8 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \end{bmatrix}$$

(5)

3a) Is the homogeneous problem  $A\mathbf{x}=\mathbf{0}$  always solvable?

3b) Is the inhomogeneous problem  $A\mathbf{x}=\mathbf{b}$  solvable no matter the choice of  $\mathbf{b}$ ?

3c) How many solutions are there? How many free variables are there? How does this number relate to the reduced row echelon form of  $A$ ?

Exercise 4) Now consider the matrix  $B$  and similar questions:

$\triangleright B := \text{Matrix}(3, 2, [1, 2, -1, 3, 4, 2]);$   
 $\text{ReducedRowEchelonForm}(B);$

$$B := \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(6)

4a) How many solutions to the homogeneous problem  $B\underline{x} = \underline{0}$ ?

4b) Is the inhomogeneous problem  $B\underline{x} = \underline{b}$  solvable for every right side vector  $\underline{b}$ ? 4c) When the inhomogeneous problem is solvable, how many solutions does it have?

Exercise 5) Square matrices (i.e number of rows equals number of columns) with 1's down the diagonal which runs from the upper left to lower right corner are special. They are called identity matrices.

**>**  $C := \text{Matrix}(4, 4, [1, 0, -1, 1, 22, -1, 3, 5, 7, 4, 6, 2, 3, 5, 7, 13]);$   
 $\text{ReducedRowEchelonForm}(C);$

$$C := \begin{bmatrix} 1 & 0 & -1 & 1 \\ 22 & -1 & 3 & 5 \\ 7 & 4 & 6 & 2 \\ 3 & 5 & 7 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(7)

**>**

5a) How many solutions to the homogeneous problem  $C\mathbf{x} = \mathbf{0}$ ?

5b) Is the inhomogeneous problem  $C\mathbf{x} = \mathbf{b}$  solvable for every choice of  $\mathbf{b}$ ?

5c) How many solutions?

**Exercise 6: What are your general conclusions?**

6a) What conditions on the reduced row echelon form of the matrix  $A$  guarantee that the homogeneous equation  $A\mathbf{x} = \mathbf{0}$  has infinitely many solutions?

6b) What conditions on the dimensions of  $A$  (i.e. number of rows and number of columns) always force infinitely many solutions to the homogeneous problem?

6c) What conditions on the reduced row echelon form of  $A$  guarantee that solutions  $\mathbf{x}$  to  $A\mathbf{x} = \mathbf{b}$  are always unique (if they exist)?

6d) If  $A$  is a square matrix ( $m=n$ ), what can you say about the solution set to  $A\mathbf{x} = \mathbf{b}$  when

- \* The reduced row echelon form of  $A$  is the identity matrix?
- \* The reduced row echelon form of  $A$  is not the identity matrix?