Math 2250-4
Tues Feb 28
5.1 Second order linear differential equations, and vector space theory connections.

In Chapter 5 we focus on the vector space
$$V = C(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \text{ s.t. } f \text{ is a continuous function}\}$$
and its subspaces.

Exercise 0) Verify that the vector space axioms for linear combinations are satisfied. Recall that the function $f + g$ is defined by $(f + g)(x) := f(x) + g(x)$ and the scalar multiple $c f(x)$ is defined by $(c f)(x) := c f(x)$.

(α) Whenever $f, g \in V$ then $f + g \in V$. (closure with respect to addition)
(β) Whenever $f \in V$ and $c \in \mathbb{R}$, then $c \cdot f \in V$. (closure with respect to scalar multiplication)

As well as:
(a) $f + g = g + f$ (commutative property)
(b) $f + (g + h) = (f + g) + h$ (associative property)
(c) $\exists\ 0 \in V$ so that $f + 0 = f$ is always true. $\leftarrow$ What is the zero vector for functions?
(d) $\forall f \in V\ \exists -f \in V$ so that $f + (-f) = 0$ (additive inverses)
(e) $c \cdot (f + g) = c \cdot f + c \cdot g$ (scalar multiplication distributes over vector addition)
(f) $(c_1 + c_2) \cdot f = c_1 \cdot f + c_2 \cdot f$ (scalar addition distributes over scalar multiplication)
(g) $c_1 \cdot (c_2 \cdot f) = (c_1 c_2) \cdot f$ (associative property)
(h) $1 \cdot f = f$, $(-1) \cdot f = -f$, $0 \cdot f = 0$ (these last two actually follow from the others).

Thus all of the concepts and vector space theorems we talked about for $\mathbb{R}^m$ and its subspaces make sense for the function vector space V and its subspaces. In particular we can talk about

- the span of a finite collection of functions $f_1, f_2, \ldots f_n$.
- linear independence/dependence for a collection of functions $f_1, f_2, \ldots f_n$.
- subspaces of V
- bases and dimension for finite dimensional subspaces. (The function space V itself is infinite dimensional, meaning that no finite collection of functions spans it.)

Definition: A second order linear differential equation for a function $y(x)$ is a differential equation that can be written in the form

$$A(x)y'' + B(x)y' + C(x)y = F(x) .$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A(x) \neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$y'' + p(x)y' + q(x)y = f(x) .$$

One reason this DE is called linear is that the operator L defined by

$$L(y) := y'' + p(x)y' + q(x)y$$

satisfies the so-called linearity properties

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(c\,y) = c\,L(y),\ c \in \mathbb{R}$.

(Recall that the matrix multiplication function $L(\underline{\mathbf{x}}) := A\,\underline{\mathbf{x}}$ satisfied the analogous properties.)

Exercise 1a) Check the linearity properties (1),(2) for the differential operator L .

1b) Use these properties to show that the solution space to the homogeneous second order linear DE

$$y'' + p(x)y' + q(x)y = 0$$

is a subspace.

1c) Use the linearity properties to show

Theorem 1: The general solution to the nonhomogeneous second order linear DE

$$y'' + p(x)y' + q(x)y = f(x)$$

is $y = y_P + y_H$ where y_P is any single particular solution and y_H is the general solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text).

Theorem 2 (Existence-Uniqueness Theorem): Let $p(x)$, $q(x)$, $f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$y'' + p(x)y' + q(x)y = f(x)$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$

and $y(x)$ exists and is twice continuously differentiable on the entire interval I.

Exercise 2) Verify Theorems 1 and 2 for the interval $I = (-\infty, \infty)$ and the IVP

$$y'' + 2y' = 3$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

Hint: This is really a first order DE for $v = y'$.

Unlike in the previous example, and unlike what was true for the first order linear differential equation

$$y' + p(x)y = q(x)$$

there is <u>not</u> a clever integrating factor formula that will always work to find the general solution of the second order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Rather, we will usually resort to vector space theory and algorithms based on clever guessing. It will help to know

<u>Theorem 3:</u> The solution space to the second order homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

is 2-dimensional.

This Theorem and the techniques we'll be using is illustrated by
<u>Exercise 3)</u> Consider the homogeneous linear DE for $y(x)$

$$y'' - 2y' - 3y = 0$$

<u>3a)</u> Find two exponential functions $y_1(x) = e^{rx}, y_2(x) = e^{\rho x}$ that solve this DE. (We actually did this computation on Feb. 17.)
<u>3b)</u> Show that every IVP

$$y'' - 2y' - 3y = 0$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

can be solved with a unique linear combination $y(x) = c_1 y_1(x) + c_2 y_2(x)$.
Then use the uniqueness theorem to deduce that y_1, y_2 are a basis for the solution space to this homogeneous differential equation.

Although we don't have the tools yet to prove the existence-uniqueness result Theorem 2, we can use it to prove the dimension result Theorem 3. Here's how (and this is really just an abstractified version of the example on the previous page):

Consider the homogeneous differential equation

$$y'' + p(x)y' + q(x)y = 0$$

on an interval I for which the hypotheses of the existence-uniqueness theorem hold.
Pick any $x_0 \in I$. Find solutions $y_1(x), y_2(x)$ to IVP's at x_0 so that the so-called Wronskian matrix for y_1, y_2 at x_0,

$$W(y_1, y_2)(x_0) = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$$

is invertible (i.e. $\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \end{bmatrix}, \begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \end{bmatrix}$ are a basis for $\mathbb{R}^2$, or equivalently so that the determinant of the Wronskian matrix (called just the Wronskian) is non-zero at x_0).

- You may be able to find suitable y_1, y_2 by good guessing, as in the previous example, but the existence-uniqueness theorem guarantees they exist even if you don't know how to find formulas for them.

Under these conditions, the solutions y_1, y_2 are actually a basis for the solution space!
•span: the condition that the Wronskian matrix is invertible at x_0 means we can solve each IVP there with a linear combination $y = c_1 y_1 + c_2 y_2$: In that case, $y' = c_1 y_1' + c_2 y_2'$ so to solve the IVP

$$\begin{aligned} y'' + p(x)y' + q(x)y &= 0 \\ y(x_0) &= b_0 \\ y'(x_0) &= b_1 \end{aligned}$$

we set

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= b_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= b_1 \end{aligned}$$

which has unique solution $[c_1, c_2]^T$ given by

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}^{-1} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

Since the uniqueness theorem says each IVP has a unique solution, this must be it. Since each solution $y(x)$ to the differential equation solves *some* initial value problem at x_0, each solution $y(x)$ *is* a linear combination of y_1, y_2.

- This shows that y_1, y_2 span the solution space.

Now, how could a linear combination $y = c_1 y_1 + c_2 y_2 \equiv 0$? In this case $y' \equiv 0$ as well, so at x_0 we would have $y(x_0) = y'(x_0) = 0$. So this solution has initial values $b_0 = b_1 = 0$. Thus $c_1 = c_2 = 0$ as well.

- This shows y_1, y_2 are linearly independent.
- Thus y_1, y_2 are a basis for the solution space, and every solution $y(x)$ can be written uniquely as

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for all $x \in I$.