

Math 2250-4

Mon Feb 27

4.4 Basis and dimension, and review of 4.1-4.4

Exercise 1) Vocabulary and concept review. Tomorrow we'll return to differential equations, and although it may seem hard to believe right now, these concepts and words will apply when we're discussing the solution spaces to differential equations, and our "vectors" are functions.

A linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is

The span of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is

The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent iff

The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly dependent iff

V is a vector space means

W is a subspace of a vector space V means

The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are a basis for a subspace W means

The dimension of a subspace W is

What are the two natural ways that subspaces of \mathbb{R}^m occur?

On Friday, we'd completed Exercise 2a below. For reference I've filled in the details we worked out. We'll work through the rest of Exercise 2 as well as Exercise 3 and Exercise 4, which are slightly reorganized from Friday's notes. Your first homework exercise this week parallels this discussion and reviews many of the key ideas in 4.1-4.4. It takes a while and some practice for these ideas to gel.

Exercise 2a) Consider the matrix equation $A \underline{x} = \underline{0}$, with the matrix A (and its reduced row echelon form) shown below. Find a basis for the solution space $W = \{\underline{x} \in \mathbb{R}^6 \text{ s.t. } A \underline{x} = \underline{0}\}$ by backsolving and writing your explicit solutions in linear combination form. Thus by construction, the vectors used to create the general linear combination will span the solution space. Discuss why they satisfy the other condition to be a basis, namely that they're linearly independent.

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We backsolved the reduced matrix (augmented with a column of zeroes because it's the homogeneous matrix equation). We got $x_6 = t \in \mathbb{R}$, $x_5 = s \in \mathbb{R}$, $x_4 = r \in \mathbb{R}$, $x_3 = -2r + s - 3t$, $x_2 = q \in \mathbb{R}$, $x_1 = -2q - r - s - 2t$. In vector linear combination form this reads

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = q \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus the four vectors on the right (from which the linear combinations are constructed) definitely span the solution space to $A \underline{x} = \underline{0}$. To show these four vectors are linearly independent, set the linear combination on the right equal to the zero vector. For each of the four linear combination coefficients (which we're calling q, r, s, t rather than c_1, c_2, c_3, c_4) deduce they must be zero by looking at the entries corresponding the column of reduced matrix which gave rise to these free parameters. Deduce $q = 0$ (entry 2), $r = 0$ (entry 4), $s = 0$ (entry 5), $t = 0$ (entry 6). Thus the four vectors are also linearly independent. Since we have verified that these four vectors span the solution space and are linearly independent, we deduce that

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Are a basis for the solution space to $A \underline{x} = \underline{0}$.

2b) What is the dimension of W ?

Finding bases for subspaces is important because it allows you to explicitly and uniquely describe each element of the subspace. You can always find a basis for the solution space to the matrix equation $A \underline{x} = \underline{0}$ just as we did in the example above.

In other subspace examples you may have a collection of vectors that span the subspace but are linearly dependent, or you may have a collection of vectors that does not yet span the entire subspace. Exercise 3 and Exercise 4 deal with how to get bases in those cases.

3a) Consider the six columns of the matrix A above, and the subspace of \mathbb{R}^4 that they span. As we've discussed before, a linear dependency

$$c_1 \text{col}_1(A) + c_2 \text{col}_2(A) + \dots + c_6 \text{col}_6(A) = \underline{0}$$

corresponds to a solution to the matrix equation $A \underline{c} = \underline{0}$. Since elementary row operations on A do not change the homogeneous equation solution set, we may therefore read off the dependencies for the 6 columns of A from the dependencies of the 6 columns of the reduced row echelon form of A . Use this information to remove "redundant" vectors from the original six, in order to explain why $\text{col}_1(A)$, $\text{col}_3(A)$ are actually a basis for the span of all six columns. Use

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

What is the dimension of the subspace of \mathbb{R}^4 spanned by the six columns of A ?

3b) There's a connection between the basis Exercise 2 and the one Exercise 3, even though in Exercise 2 we were looking at a subspace of \mathbb{R}^6 and in Exercise 3 we're looking at a subspace of \mathbb{R}^4 . The connection is that homogeneous solutions \underline{c} to $A \underline{c} = \underline{0}$ correspond to column dependencies for A . Check this correspondence, by looking once more at the basis vectors for the solution space to $A \underline{c} = 0$ that you found in Exercise 2 and the column dependencies you found in 3a. Here were the Exercise 2 basis vectors:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

3c) This connection between homogeneous solutions and column dependencies is the reason why each matrix B has exactly one reduced row echelon form: The four conditions for reduced row echelon form,

- zero rows at bottom,
- non-zero rows start with leading ones,
- leading ones in successive rows move to the right,
- all column entries in any column containing a leading "1" are zero except for that "1"),

together with specified column dependencies uniquely determine the rref matrix. Discuss this for the matrix A and its reduced row echelon form:

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- As Exercise 3 illustrates, if we have a spanning set for a subspace W , then we can remove dependent vectors from this set without shrinking the span, so that the remaining vectors will still span W . After a finite number of steps we'll remove enough vectors to make the remaining ones linearly independent, i.e. a basis for W .

- A dual problem that could arise when trying to find a basis for W is that you might have a set of independent vectors in W , but they don't yet span all of W . In this case, if you augment your set with a vector not already in the span, then the larger set will remain independent*. Repeating this step a finite number of times, you can build up to a basis.

* Check: If $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ are linearly independent, and $\underline{w} \notin \text{span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$, then $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n, \underline{w}$ are also linearly independent: Let

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots + c_n \underline{v}_n + d \underline{w} = \underline{0}.$$

Since $\underline{w} \notin \text{span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ it must be that $d = 0$ (since otherwise we could solve for \underline{w} as a linear combination of $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$). But once $d = 0$, we have

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots + c_n \underline{v}_n = \underline{0}$$

which implies $c_1 = c_2 = \dots = c_n = 0$ by the independence of $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$.

Exercise 4) Consider the first and third columns of A , which are a basis for the 2-dimensional subspace of \mathbb{R}^4 spanned by all six columns of A . Find two more vectors to augment with these two, in order to get a basis for all of \mathbb{R}^4 containing these two vectors. Hint: the standard basis vectors of \mathbb{R}^4 , namely

$$e_1 = [1, 0, 0, 0]^T, e_2 = [0, 1, 0, 0]^T, e_3 = [0, 0, 1, 0]^T, e_4 = [0, 0, 0, 1]^T$$

span \mathbb{R}^4 , so clearly at least two of these can't be in the span of the two vectors we're starting with. You can pick two of these by using a rref computation:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & -1 & \frac{3}{2} \end{bmatrix}$$

Some important facts about spanning sets, independence, bases and dimension follow from one key fact, and then logic:

key fact: If n vectors $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ span a subspace W then any collection $\underline{w}_1, \underline{w}_2, \dots, \underline{w}_N$ of vectors in W with $N > n$ will always be linearly dependent. (This is explained on pages 254-255 of the text. Notice that it fits our intuition based on what we know in the special case that $W = \mathbb{R}^n$.)

Thus:

1) If a finite collection of vectors in W is linearly independent, then no collection with fewer vectors can span all of W . (This is because if the smaller collection did span, the larger collection wouldn't have been linearly independent after all, by the key fact.)

2) Every basis of W has the same number of vectors, so the concept of dimension is well-defined and doesn't depend on choice of basis. (This is because if $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ are a basis for W then every larger collection of vectors is dependent by the key fact and every smaller collection fails to span by (1), so only collections with exactly n vectors have a chance to be bases.)

3) If the dimension of W is the number n , and if $\underline{w}_1, \underline{w}_2, \dots, \underline{w}_n$ span W then they're automatically linearly independent and thus a basis. (If they were dependent we could delete at least one of them and still have a spanning set, which would violate (1).)

4) If the dimension of W is the number n , and if $\underline{w}_1, \underline{w}_2, \dots, \underline{w}_n$ are in W and are linearly independent, then they automatically span W and thus are a basis. (If they didn't span W we could augment with a vector \underline{w}_{n+1} not in their span and have a collection of $n + 1$ independent vectors in W , violating the key fact.)