Math 2250-4
Fri Feb 24
4.4 Basis and dimension

Exercise 1) Vocabulary review:

A <u>linear combination</u> of the vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ is

The <u>span</u> of $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ is

The vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ are <u>linearly independent</u> iff

The vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ are <u>linearly dependent</u> iff

V is a <u>vector space</u> means

W is a <u>subspace</u> of a vector space V means

• New concepts for today:

<u>Definition:</u> The vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ are called a <u>basis</u> for a subspace W iff they <u>span</u> W and are <u>linearly independent</u>. (We talked about a basis for the span of a collection of vectors before, which this definition generalizes.)

<u>Definition:</u> The <u>dimension</u> of a subspace W is the number of vectors in a basis for W. (It turns out that all bases for a subspace always have the same number of vectors.)

• Finish <u>Exercises 6-8</u> from Wednesday's notes.

On Wednesday we talked about the fact that subspaces arise naturally in two ways:

(1) As the span of a collection of vectors,
$$W = span\{\underline{v}_1, \underline{v}_2,... \underline{v}_n\} .$$
(2) As the solution space to a homogenous matrix equation,
$$W = \{\underline{x} \in \mathbb{R}^n \text{ s.t. } A_{m \times n} \underline{x} = \underline{0}\}$$

It turns out that our algorithm for solving homogeneous matrix equations via reduced row echelon form and backsolving automatically yields bases for the solution space:

<u>Exercise 2)</u> Consider the matrix equation $A\underline{x} = \underline{0}$, with the matrix A (and its reduced row echelon form) shown below:

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2a) Find a basis for the solution space $W = \{\underline{x} \in \mathbb{R}^6 \text{ s.t. } A\underline{x} = \underline{0}\}$ by backsolving and writing your explicit solutions in linear combination form. Thus by construction, the vectors used to create the general linear combination will span the solution space. Discuss why they satisfy the other condition to be a basis, namely that they're linearly independent.

2b) What is the dimension of W?

2c) Since solutions to homogeneous matrix equations are exactly linear dependencies between the columns of the matrix, your basis in <u>2a</u> can be thought of as a "basis" for all the column dependencies. Check this.

(By the way, this connection between homogeneous solutions and column dependencies is the reason why any matrix has only one reduced row echelon form - the four conditions for rref (zero rows at bottom, non-zero rows start with leading ones, leading ones in successive rows move to the right, all column entries in any column containing a leading "1" are zero except for that "1"), together with specified column dependencies uniquely determine the rref matrix.)

Finding bases for subspaces is important.

• We've already figured out that if we have a spanning set for W, then we can remove dependent vectors from this set without shrinking the span, so that the remaining vectors will still span W. After a finite number of steps we'll remove enough vectors to make the remaining ones linearly independent, i.e. a basis for W.

Exercise 3) Why are the first and third columns of the matrix A in Exercise 2 a basis for the span of all six columns?

• A dual problem that could arise when trying to find a basis for W is that you have a set of independent vectors in W, but they don't yet span all of W. In this case, if you augment your set with a vector not already in the span, then the larger set will remain independent*. Repeating this step a finite number of times, you can build up to a basis.

* Check: If $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ are linearly independent, and $\underline{w} \notin span\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n\}$, then $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n, \underline{w}$ are also linearly independent: Let
$$c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_n\underline{v}_n + d\,\underline{w} = \underline{0}.$$
Since $\underline{w} \notin span\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n\}$ it must be that $d = 0$ (since otherwise we could solve for $\underline{w}$ as a linear combination of $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$). But once $d = 0$, we have
$$c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_n\underline{v} = \underline{0}$$
which implies $c_1 = c_2 = \ldots = c_n = 0$ by the independence of $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$.

Exercise 4) Consider the first and third columns of A, which are a basis for the 2-dimensional subspace of $\mathbb{R}^4$ spanned by all six columns of A. Find two more vectors to augment with these two, in order to get a basis for all of $\mathbb{R}^4$ containing these two vectors. Hint: the standard basis vectors of $\mathbb{R}^4$, namely

$$e_1 = [1, 0, 0, 0]^T, e_2 = [0, 1, 0, 0]^T, e_3 = [0, 0, 1, 0]^T, e_4 = [0, 0, 0, 1]^T$$

span $\mathbb{R}^4$, so clearly at least two of these can't be in the span of the two vectors we're starting with. You can pick two of these by using a rref computation:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -\dfrac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & -\dfrac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \dfrac{1}{2} \\ 0 & 0 & 0 & 1 & -1 & \dfrac{3}{2} \end{bmatrix}$$