Math 2250-4 homework problem session today 3:30-4:30 LCB 215
Tues Feb 14 old exam work-through today 5:00-6:30 AEB 310

3.6 Determinants and linear systems of equations

First, finish discussing how elementary row operations affect determinants, and why determinants determine whether or not matrix inverses exist, pages 4-6 of Monday's notes.

Then, in order to understand the magic formula for matrix inverses, we first need to talk about matrix *transposes:*

<u>Definition:</u> Let $B_{m \times n} = [b_{i\,j}]$. Then the <u>transpose</u> of B, denoted by B^T is an $n \times m$ matrix defined by

$$entry_{i\,j}(B^T) := entry_{j\,i}(B) = b_{j\,i}\,.$$

The effect of this definition is to turn the columns of B into the rows of B^T :

$$entry_i\left(col_j(B)\right) = b_{i\,j}\,.$$
$$entry_i\left(row_j(B^T)\right) = entry_{j\,i}(B^T) = b_{i\,j}\,.$$

And to turn the rows of B into the columns of B^T:

$$entry_j\left(row_i(B)\right) = b_{i\,j}$$
$$entry_j\left(col_i(B^T)\right) = entry_{j\,i}(B^T) = b_{i\,j}\,.$$

<u>Exercise 1)</u> explore these properties with the identity

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

<u>Theorem:</u> Let $A_{n \times n}$, and denote its <u>cofactor matrix</u> by $cof(A) = [C_{i j}]$, with $C_{i j} = (-1)^{i+j} M_{i j}$, and $M_{i j} =$ the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A. Define the <u>adjoint matrix</u> to be the transpose of the cofactor matrix:

$$Adj(A) := cof(A)^T$$

Then, when A^{-1} exists it is given by the formula

$$A^{-1} = \frac{1}{det(A)} Adj(A) .$$

<u>Exercise 2)</u> Show that in the 2×2 case this reproduces the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} .$$

<u>Exercise 3)</u> Yesterday, for our friend $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$ we worked out $cof(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$ and $det(A) = 15$. Use the Theorem to find A^{-1} and check your work. Does the matrix multiplication relate to the dot products we computed yesterday?

Exercise 4) Continuing with our example,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad cof(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix} \qquad Adj(A) = \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

4a) The $(1, 1)$ entry of $(A)(Adj(A))$ is $15 = 1\cdot5 + 2\cdot2 + (-1)(-6)$. Explain why this is $det(A)$, expanded across the first row.

4b) The $(2, 1)$ entry of $(A)(Adj(A))$ is $0\cdot5 + 3\cdot2 + (1)(-6) = 0$. Notice that you're using the same cofactors as in (4a). What matrix, which is obtained from A by keeping two of the rows, but replacing a third one with one of those two, is this the determinant of?

4c) The $(3, 2)$ entry of $(A)(Adj(A))$ is $2\cdot0 - 2\cdot3 + 1\cdot6 = 0$. What matrix (which uses two rows of A) is this the determinant of?

If you completely understand 4abc, then you have realized why

$$(A)(Adj(A)) = det(A)I$$

for every square matrix, and so also why

$$A^{-1} = \frac{1}{det(A)} Adj(A).$$

Precisely,

$$entry_{i\,i}\, A(Adj(A)) = row_i(A) \cdot col_i(Adj(A)) = row_i(A) \cdot row_i(cof(A)) = det(A),$$

expanded across the i^{th} row.

On the other hand, for $i \neq k$,

$$entry_{k\,i}\, A(Adj(A)) = row_k(A) \cdot col_i(Adj(A)) = row_k(A) \cdot row_i(cof(A)) = 0$$

because it is the determinant of a matrix made from A by replacing the i^{th} row with the k^{th} row, and when two rows are equal, the determinant of any matrix is zero.

There's a related formula for solving for individual components of $\underline{\boldsymbol{x}}$ when $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ has a unique solution ($\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$). This can be useful if you only need one or two components of the solution vector, rather than all of it:

Cramer's Rule: Let $\underline{\boldsymbol{x}}$ solve $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$, for invertible A. Then

$$x_k = \frac{det(A_k)}{det(A)}$$

where A_k is the matrix obtained from A by replacing the k^{th} column with $\underline{\boldsymbol{b}}$.

proof: Since $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$ the k^{th} component is given by

$$\begin{aligned} x_k &= entry_k\left(A^{-1}\underline{\boldsymbol{b}}\right) \\ &= entry_k\left(\frac{1}{|A|}Adj(A)\underline{\boldsymbol{b}}\right) \\ &= \frac{1}{|A|}row_k(Adj(A)) \cdot \underline{\boldsymbol{b}} \\ &= \frac{1}{|A|}col_k(cof(A)) \cdot \underline{\boldsymbol{b}}. \end{aligned}$$

Notice that $col_k(cof(A)) \cdot \underline{\boldsymbol{b}}$ is the determinant of the matrix obtained from A by replacing the k^{th} column by $\underline{\boldsymbol{b}}$, where we've computed that determinant by expanding down the k^{th} column! This proves the result.

Exercise 5) Solve $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

5a) With Cramer's rule

5b) With A^{-1}, using the adjoint formula.