

Recall, at the end of class on Friday we discussed a magic formula for the inverse of 2×2 matrices, which involved the *determinant*:

Theorem: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1}$ exists if and only if the *determinant* $D := a_{11}a_{22} - a_{21}a_{12}$ of $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is non-zero. And in this case,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{D} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

(Notice that the diagonal entries have been swapped, and minus signs have been placed in front of the off-diagonal terms. This formula should be memorized.)

We checked that this formula for the inverse works, for $D \neq 0$. We know that matrices have inverses if and only if their reduced row echelon form is the identity. We can check that 2×2 matrices reduce to the identity if and only if their two rows are not multiples of each other. Then we can check that the inequality $a_{11}a_{22} - a_{21}a_{12} \neq 0$ is exactly the condition that the two rows are not multiples of each other. For example, if the entries in the top row are both non-zero then to say that row 2 is not a multiple of row one is to say that $\frac{a_{21}}{a_{11}} \neq \frac{a_{22}}{a_{12}}$ which is the same as the determinant condition. The remaining cases are similar.

There is a similar development for determinants of $n \times n$ matrices, with $n > 2$. These determinants are defined recursively:

Definition: Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A , written $\det(A)$ or $|A|$, is defined by

$$\det(A) := \sum_{j=1}^n a_{1j}(-1)^{1+j}M_{1j} = \sum_{j=1}^n a_{1j}C_{1j}.$$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j}M_{1j}$.

More generally, the determinant of the $(n - 1) \times (n - 1)$ matrix obtained by deleting row i and column j from A is called the ij Minor M_{ij} of A , and $C_{ij} := (-1)^{i+j} M_{ij}$ is called the ij Cofactor of A .

Theorem: (proof is in text appendix) $\det(A)$ can be computed by expanding across any row, say row i :

$$\det(A) := \sum_{j=1}^n a_{ij} (-1)^{i+j} M_{ij} = \sum_{j=1}^n a_{ij} C_{ij}$$

or by expanding down any column, say column j :

$$\det(A) := \sum_{i=1}^n a_{ij} (-1)^{i+j} M_{ij} = \sum_{i=1}^n a_{ij} C_{ij}.$$

Exercise 1a) Let $A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$. Compute $\det(A)$ using the definition.

1b) Verify that the matrix of all the cofactors of A is given by $[C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$. Then expand

$\det(A)$ down various columns and rows using the a_{ij} factors and C_{ij} cofactors. Verify that you always get the same value for $\det(A)$. Notice that in each case you are taking the dot product of a row (or column) of A with the corresponding row (or column) of the cofactor matrix.

1c) What happens if you take dot products between a row of A and a *different* row of $[C_{ij}]$? A column of A and a *different* column of $[C_{ij}]$?

Exercise 2) Compute the following determinants by being clever about which rows or columns to use:

$$\underline{2a)} \begin{vmatrix} 1 & 38 & 106 & 3 \\ 0 & 2 & 92 & -72 \\ 0 & 0 & 3 & 45 \\ 0 & 0 & 0 & -2 \end{vmatrix};$$

$$\underline{2b)} \begin{vmatrix} 1 & 0 & 0 & 0 \\ \pi^2 & 2 & 0 & 0 \\ 0.476 & 88 & 3 & 0 \\ 1 & 22 & 33 & -2 \end{vmatrix}.$$

Exercise 3) Explain why it is always true that for an upper triangular matrix (as in 2a), or for a lower triangular matrix (as in 2b), the determinant is always just the product of the diagonal entries.

The effective way to compute determinants for larger-sized matrices without lots of zeroes is to use the following facts, which track how elementary row operations affect determinants:

- (1a) Swapping any two rows changes the sign of the determinant.

proof: This is clear for 2×2 matrices, since

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad.$$

For 3×3 determinants, expand across the row *not* being swapped, and use the 2×2 swap property to deduce the result. Prove the general result by induction: once it's true for $n \times n$ matrices you can prove it for any $(n + 1) \times (n + 1)$ matrix, by expanding across a row that wasn't swapped, and applying the $n \times n$ result.

- (1b) Thus, if two rows in a matrix are the same, the determinant of the matrix must be zero: on the one hand, swapping those two rows leaves the matrix and its determinant unchanged; on the other hand, by (1a) the determinant changes its sign. The only way this is possible is if the determinant is zero.

- (2a) If you factor a constant out of a row, then you factor the same constant out of the determinant.

Precisely, using \mathcal{R}_i for i^{th} row of A , and writing $\mathcal{R}_i = c \mathcal{R}_i^*$

$$\begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i \\ \vdots \\ \mathcal{R}_n \end{vmatrix} = \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ c \mathcal{R}_i^* \\ \vdots \\ \mathcal{R}_n \end{vmatrix} = c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i^* \\ \vdots \\ \mathcal{R}_n \end{vmatrix}.$$

proof: expand across the i^{th} row, noting that the corresponding cofactors don't change, since they're computed by deleting the i^{th} row:

$$\det(A) = \sum_{j=1}^n a_{ij} C_{ij} = \sum_{j=1}^n c a_{ij}^* C_{ij} = c \sum_{j=1}^n a_{ij}^* C_{ij} = c \det(A^*).$$

- (2b) Combining (2a) with (1b), we see that if one row in A is a scalar multiple of another, then $\det(A) = 0$.

- (3) If you replace row i of A by its sum with a multiple of another row, then the determinant is unchanged! Expand across the i^{th} row:

$$\begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_k \\ \mathcal{R}_i + c \mathcal{R}_k \\ \vdots \\ \mathcal{R}_n \end{vmatrix} = \sum_{j=1}^n (a_{ij} + c a_{kj}) C_{ij} = \sum_{j=1}^n a_{ij} C_{ij} + c \sum_{j=1}^n a_{kj} C_{ij} = \det(A) + c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_k \\ \mathcal{R}_k \\ \vdots \\ \mathcal{R}_n \end{vmatrix} = \det(A).$$

Remark: The analogous properties hold for corresponding "elementary column operations". In fact, the proofs are almost identical, except you use column expansions.

Exercise 4) Recompute $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$ using elementary row operations (and/or elementary column operations).

Exercise 5) Compute $\begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ -1 & 0 & -2 & 1 \end{vmatrix}$.

Maple check:

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[> with(LinearAlgebra) :
> A := Matrix(4, 4, [1, 0, -1, 2, 2, 1, 1, 0, 2, 0, 1, 1, -1, 0, -2, 1]);
> Determinant(A);
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Tomorrow we'll find the magic formula for matrix inverses that uses determinants. But today we have the tools to check that the determinant does determine whether or not matrices have inverses:

Theorem: Let $A_{n \times n}$. Then A^{-1} exists if and only if $\det(A) \neq 0$.

proof: We already know that A^{-1} exists if and only if the reduced row echelon form of A is the identity matrix. Now, consider reducing A to its reduced row echelon form, and keep track of how the determinants of the corresponding matrices change: As we do elementary row operations,

- if we swap rows, the sign of the determinant switches.
- if we factor non-zero factors out of rows, we factor the same factors out of the determinants.
- if we replace a row by its sum with a multiple of another row, the determinant is unchanged.

Thus,

$$|A| = c_1 |A_1| = c_1 c_2 |A_2| = \dots = c_1 c_2 \dots c_N |rref(A)|$$

where the nonzero c_k 's arise from the three types of elementary row operations. If $rref(A) = I$ its determinant is 1, and $|A| = c_1 c_2 \dots c_N \neq 0$. If $rref(A) \neq I$ then its bottom row is all zeroes and its determinant is zero, so $|A| = c_1 c_2 \dots c_N (0) = 0$. Thus $|A| \neq 0$ if and only if $rref(A) = I$ if and only if A^{-1} exists.

Remark: Using the same ideas as above, you can show that $\det(AB) = \det(A)\det(B)$. This is an important identity that gets used, for example, in multivariable change of variables formulas for integration, using the Jacobian matrix. (It is not true that $\det(A+B) = \det(A) + \det(B)$.) Here's how to show $\det(AB) = \det(A)\det(B)$: The key point is that if you do an elementary row operation to AB , that's the same as doing the elementary row operation to A , and then multiplying by B . With that in mind, if you do exactly the same elementary row operations as you did for A in the theorem above, you get

$$|AB| = c_1 |A_1 B| = c_1 c_2 |A_2 B| = \dots = c_1 c_2 \dots c_N |rref(A)B|.$$

If $rref(A) = I$, then from the theorem above, $|A| = c_1 c_2 \dots c_N$, and we deduce $|AB| = |A||B|$. If

$rref(A) \neq I$, then its bottom row is zeroes, and so is the bottom row of $rref(A)B$. Thus $|AB| = 0$ and also $|A||B| = 0$.