Math 2250-4
Fri Feb 10
3.5 Matrix inverses.

Recall,

Matrix inverses: A square matrix $A_{n \times n}$ is <u>invertible</u> if there is a matrix $B_{n \times n}$ so that
$$AB = BA = I \,.$$
In this case we call B the inverse of A, and write $B = A^{-1}$. A matrix can have at most one inverse matrix, and we see from the above that $\left(A^{-1} \right)^{-1} = A$.

On Wednesday we checked that for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} -2 & 1 \\ \dfrac{3}{2} & -\dfrac{1}{2} \end{bmatrix}.$$

We were discussing one way in which inverse matrices are useful:

Theorem: If A^{-1} exists then the only solution to $A\underline{x} = \underline{b}$ is $\underline{x} = A^{-1}\underline{b}$.

We checked that the theorem worked in the case of the system
$$\begin{array}{c} x + 2\, y = 5 \\ 3\, x + 4\, y = 6 \end{array} \quad :$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \dfrac{3}{2} & -\dfrac{1}{2} \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 \\ \dfrac{9}{2} \end{bmatrix}.$$

<u>Exercise 1)</u> Use matrix algebra to verify why the Theorem is true. Notice that the correct formula is $\underline{x} = A^{-1}\underline{b}$ and not $\underline{x} = \underline{b}\, A^{-1}$ (This second product can't even be computed! Because matrix multiplication is not commutative it is very important to be careful with the order in which matrix products are written down).

But where did that formula for A^{-1} come from?

Answer: Consider A^{-1} as an unknown matrix, $A^{-1} = X$. We want
$$A X = I.$$
In the 2×2 case, we can break this matrix equation down by columns:
$$A \left[col_1(X) \mid col_2(X) \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$
We've solved several linear systems for different right hand sides but the same coefficient matrix before, so

let's do it again: the first column of $X = A^{-1}$ should solve $A\,\underline{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the second column should solve

$A\,\underline{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

Exercise 2: Reduce the double augmented matrix
$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix}$$
to find the two columns of A^{-1} for the previous example.

Exercise 3) Will this always work? Can you find A^{-1} for
$$A := \begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix} ?$$

Exercise 4) Will this always work? Try to find B^{-1} for $B := \begin{bmatrix} 1 & 5 & 5 \\ 2 & 5 & 0 \\ 2 & 7 & 4 \end{bmatrix}$.

Hint: We'll discover that it's impossible for B to have an inverse.

Exercise 5) What happens when we ask software like Maple for the inverse matrices above?

> $with(LinearAlgebra)$:

> $\begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix}^{-1}$;

> $\begin{bmatrix} 1 & 5 & 5 \\ 2 & 5 & 0 \\ 2 & 7 & 4 \end{bmatrix}^{-1}$;

<u>Theorem:</u> Let $A_{n \times n}$ be a square matrix. Then A has an inverse matrix if and only if its reduced row echelon form is the identity. In this case the algorithm illustrated on the previous page will always yield the inverse matrix.

<u>explanation:</u> By the previous theorem, when A^{-1} exists, the solutions to linear systems
$$A\underline{x} = \underline{b}$$
are unique $(\underline{x} = A^{-1}\underline{b})$. From previous discussion we know that for square matrices, solutions to such linear systems exist and are unique if and only if the reduced row echelon form of A is the identity. (Do you remember why?) Thus A^{-1} can only possibly exist in the case that A does reduce to the identity.

In this case that A does reduce to I, we search for A^{-1} as the solution matrix X to the matrix equation
$$A\,X = I$$
i.e.

$$\left[A \;\middle|\; col_1(X) \;\middle|\; col_2(X) \;\middle| \;....\; \middle|\; col_n(X) \right] = \left[\begin{array}{cc|cc} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & .... & 0 \\ 0 & 0 & & 1 \end{array} \right]$$

Because A reduces to the identity matrix, we may use a chain of elementary row operations:

$$[\,A \,|\, I\,] \to \to \to \to \to [\,I \,|\, B\,],$$
and deduce that the columns of X are exactly the columns of B, i.e. $X = B$. Thus we know that
$$A\,B = I\,.$$
To realize that $B\,A = I$ as well, we would try to solve $B\,Y = I$ for Y, and hope $Y = A$. But we can do this by reordering our columns above and reversing each of the elementary row operations in the first computation, i.e.
$$[\,B \,|\, I\,] \to \to \to \to \to [\,I \,|\, A\,]\,,$$
so $B\,A = I$ also holds.

There's a nice formula for the inverses of 2 × 2 matrices, and it turns out this formula will lead to the next text section 3.6 on determinants:

Theorem: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ exists if and only if the <u>determinant</u> D = $ad - bd$ of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-zero. And in this case,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Notice that the diagonal entries have been swapped, and minus signs have been placed in front of the off-diagonal terms. This formula should be memorized.)

<u>Exercise 6a)</u> Check that this formula for the inverse works, for D $\neq$ 0 . (We could have derived it with elementary row operations, but it's easy to check since we've been handed the formula.)

<u>6b)</u> Even with systems of two equations in two unknowns, unless they come from very special problems the algebra is likely to be messier than you might expect (without the formula above). Use the magic formula to solve the system

$$3\,x + 7\,y = 5$$
$$5\,x + 4\,y = 8\ .$$

<u>Remark:</u> For a 2 × 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the reduced row echelon form will be the identity if and only if the two rows are not multiples of each other. If a, b are both non-zero this is equivalent to saying that the ratio of the first entries in the rows $\frac{c}{a} \neq \frac{d}{b}$, the ratio of the second entries. Cross multiplying we see this is the same as $ad \neq bc$, i.e. $ad - bc \neq 0$. This is also the correct condition for the rows not being multiples, even if one or both of a, b are zero, and so by the previous theorem this is the correct condition for knowing the inverse matrix exists.

<u>Remark:</u> Determinants are defined for square matrices $A_{n \times n}$ and they determine whether or not the inverse matrices exist, (i.e. whether the reduced row echelon form of A is the identity matrix).And in these cases there are analogous (more complicated) magic formulas for the inverse matrices. This is section 3.6 material that we'll discuss carefully on Monday and Tuesday.