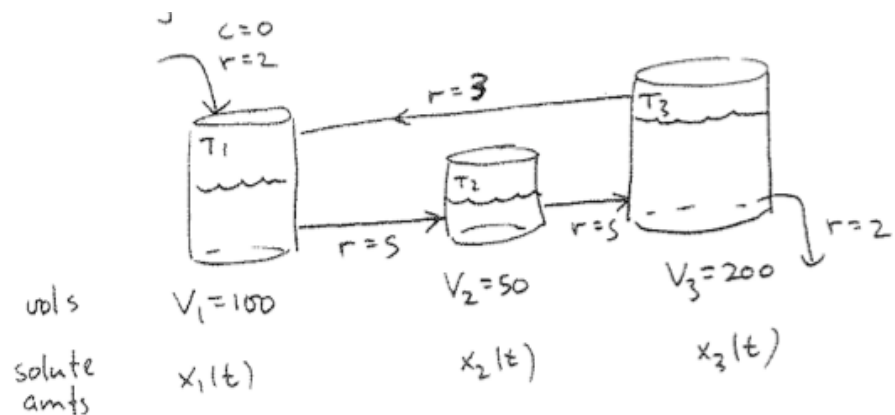


3.1-3.2 Linear systems of (algebraic) equations and how to solve them

We're going to temporarily leave differential equations in order to study basic concepts in linear algebra. You've all studied linear systems of equations and matrices before, and that's where we'll start. Linear algebra is foundational for many different disciplines, and in this course we'll use the key ideas when we return to higher order linear differential equations and to systems of differential equations. The following is just one example of the kinds of things we'll be able to do later in this course, using linear algebra concepts and computations:

Exercise 1: Consider the following configuration of three tanks, with indicated liquid volumes and pipe rates.

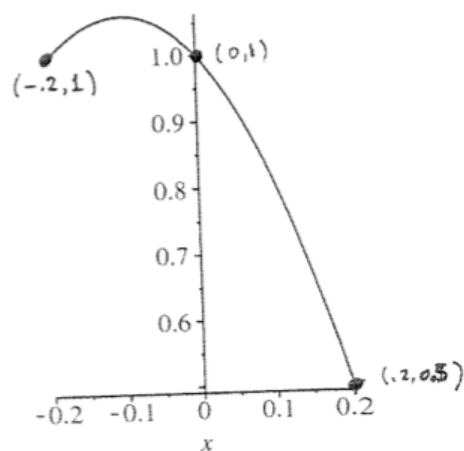
- Derive a system of three first order differential equations for the solute amounts $x_1(t)$, $x_2(t)$, $x_3(t)$ in the respective tanks.
- Rewrite this system in matrix-vector form.
- For what sort of initial value problem would you expect there to be a unique solution?



Exercise 2: Set up a system of three linear algebraic equations for the coefficients a, b, c of a quadratic function $p(x) = ax^2 + bx + c$ so that for some fixed $h > 0$ the graph $y = p(x)$ goes through the three points

$$(-h, y_{-1}), (0, y_0), (h, y_1)$$

(See example of what you're trying to do, below....this problem relates to Simpson's rule for numerical integration, see 2250-4 hw.)



$$(-.2, 1), (0, 1), (.2, .5)$$

$$p(x) = 1 - \frac{5}{4}x - \frac{25}{4}x^2 !$$

$$h = .2$$

$$y_{-1} = 1, y_0 = 1, y_1 = .5$$

In 3.1-3.2 our goal is to understand systematic ways to solve simultaneous linear equations.

- We'll often call the unknowns x_1, x_2, \dots, x_n , or write them as elements in a vector

$$\underline{\mathbf{x}} = [x_1, x_2, \dots, x_n].$$

- Then the general linear system (LS) of m equations in the n unknowns can be written as

$$\begin{array}{ccccccc} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n & = & b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n & = & b_m \end{array}$$

where the coefficients a_{ij} and the right-side number b_j are known. The goal is to find values for the vector $\underline{\mathbf{x}}$ so that all equations are true. (Thus this is often called finding "simultaneous" solutions to the linear system, because all equations will be true at once.)

Notice that we use two subscripts for the coefficients a_{ij} and that the first one indicates which equation it appears in, and the second one indicates which variable it's multiplying; in the corresponding *coefficient matrix* A , this numbering corresponds to the row and column of a_{ij} :

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Let's start small, where geometric reasoning will help us understand what's going on:

Exercise 3: Describe the solution set of each single equation below; describe and sketch its geometric realization in the indicated Euclidean spaces.

3a) $3x = 5$, for $x \in \mathbb{R}$.

3b) $2x + 3y = 6$, for $[x, y] \in \mathbb{R}^2$.

3c) $2x + 3y + 4z = 12$, for $[x, y, z] \in \mathbb{R}^3$.

2 linear equations in 2 unknowns:

$$a_{11} x + a_{12} y = b_1$$

$$a_{21} x + a_{22} y = b_2$$

goal: find all $[x, y]$ making both of these equations true. So geometrically you can interpret this problem as looking for the intersection of two lines.

Exercise 4: Consider the system of two equations E_1, E_2 :

$$E_1 \quad 5x + 3y = 1$$

$$E_2 \quad x - 2y = 8$$

4a) Sketch the solution set in \mathbb{R}^2 , as the point of intersection between two lines.

4b) Use the following three "elementary equation operations" to systematically reduce the system E_1, E_2 to an equivalent system (i.e. one that has the same solution set), but of the form

$$1x + 0y = c_1$$

$$0x + 1y = c_2$$

(so that the solution is $x = c_1, y = c_2$). Make sketches of the intersecting lines, at each stage.

The three types of elementary equation operation are below. Can you explain why the solution set to the modified system is the same as the solution set before you make the modification?

- interchange the order of the equations
- multiply one of the equations by a non-zero constant
- replace an equation with its sum with a multiple of a different equation.

4c) Look at your work in 4b. Notice that you could have save a lot of writing by doing this computation "synthetically", i.e. by just keeping track of the coefficients and right-side values. Using R_1, R_2 as symbols for the rows, your work might look like the computation below. Notice that when you operate synthetically the "elementary equation operations" correspond to "elementary row operations":

- interchange two rows
- multiply a row by a non-zero number
- replace a row by its sum with a multiple of another row.

$$\begin{array}{r|l}
 \begin{array}{cc|c}
 5 & 3 & 1 \\
 1 & -2 & 8
 \end{array} & \\
 \hline
 R_2 & \begin{array}{cc|c}
 1 & -2 & 8
 \end{array} & \\
 R_1 & \begin{array}{cc|c}
 5 & 3 & 1
 \end{array} & \\
 \hline
 & \begin{array}{cc|c}
 1 & -2 & 8
 \end{array} & \\
 -5R_1 + R_2 & \begin{array}{cc|c}
 0 & 13 & -39
 \end{array} & \\
 \hline
 & \begin{array}{cc|c}
 1 & -2 & 8
 \end{array} & \\
 & \begin{array}{cc|c}
 0 & 1 & -3
 \end{array} & \\
 \hline
 2R_2 + R_1 & \begin{array}{cc|c}
 1 & 0 & 2 \\
 0 & 1 & -3
 \end{array} & \rightarrow \begin{array}{l} x = 2 \\ y = -3 \end{array}
 \end{array}$$

4d) What are the possible geometric solutions sets to 1, 2, 3, 4 or any number of linear equations in two unknowns?

Solutions to linear equations in 3 unknowns:

What is the geometric question you're answering?

Exercise 5) Consider the system

$$\begin{aligned}x + 2y + z &= 4 \\3x + 8y + 7z &= 20 \\2x + 7y + 9z &= 23.\end{aligned}$$

Use elementary equation operations (or if you prefer, elementary row operations in the synthetic version) to find the solution set to this system. There's a systematic way to do this, which we'll talk about. It's called Gaussian elimination.

Hint: The solution set is a single point, $[x, y, z] = [5, -2, 3]$.

Exercise 6 There are other possibilities. In the two systems below we kept all of the coefficients the same as in Exercise 5, except for a_{33} , and we changed the right side in the third equation, for 6a. Work out what happens in each case.

6a)

$$\begin{aligned}x + 2y + z &= 4 \\3x + 8y + 7z &= 20 \\2x + 7y + 8z &= 20.\end{aligned}$$

6b)

$$\begin{aligned}x + 2y + z &= 4 \\3x + 8y + 7z &= 20 \\2x + 7y + 8z &= 23.\end{aligned}$$

6c) What are the possible solution sets (and geometric configurations) for 1, 2, 3, 4,... equations in 3 unknowns?