Name ____

Student I.D. ___________________________________

Math 2250-4
Exam 2 SOLUTIONS
March 29, 2012

Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. **Good Luck!**

1) Are $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix}$ a basis for $\mathbb{R}^3$? If they are, explain why. If they aren't a basis for $\mathbb{R}^3$, determine what sort of subspace of $\mathbb{R}^3$ they do span, and find a basis for that subspace.

(20 points)

Method 1) *We know that we can use the determinant to check whether n vectors in $\mathbb{R}^n$ are a basis, and indeed*

$$det\begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & 9 \\ 0 & 1 & 1 \end{bmatrix} = 0$$

so these three vectors neither span $\mathbb{R}^3$ nor are they linearly independent. Thus they are not a basis for $\mathbb{R}^3$. Since the first two vectors are linearly independent, the span of the three vectors is at least two dimensional. Since they don't span a 3-dimensional space, they must therefore span a 2-dimensional space, i.e. a plane through the origin. And so, the third vector must be a linear combination of the first two (independent vectors). Thus we may take

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

as a basis for this plane.

Method 2) *(More concrete). Reduce the matrix which has these three vectors as columns:*

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & 9 \\ 0 & 1 & 1 \end{bmatrix} ; \quad -2R_1 + R_2 \rightarrow R_2 :$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 7 & 7 \\ 0 & 1 & 1 \end{bmatrix} ; \quad \frac{1}{7}R_2 \rightarrow R_2 : \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad -R_2 + R_3 \rightarrow R_3 : \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} ;$$

$$2\,R_2 + R_1 \rightarrow R_1 : \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} .$$

Thus, since column dependencies of row equivalent matrices are the same, we deduce that

$$3\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix}.$$

This shows that $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix}$ *are dependent. Deleting any one of them will not shrink the span (because this dependency equation implies that we can express any one of deleted vectors as a linear combination of the other two). The remaining two vectors will always be linearly independent, since e.g. that is true for any two columns of the reduced matrix. Thus we can choose any two of the three original vectors to be a basis of our* <u>*plane through the origin*</u>.

2) Consider the following mechanical oscillation (or electrical circuit) differential equations. In each case answer the following questions:

(i) What is the undetermined coefficients "guess" for the particular solution $x_P(t)$? (Do NOT try to find the precise particular solution, just its form.)
(ii) What is the complete general solution to the non-homogeneous differential equation, including your undetermined coefficients particular solution?
(ii) What physical phenomenon will be exhibited by the general solutions to this differential equation?

2a) $x''(t) + 5\,x(t) = 3\cos(2\,t)$.

(8 points)

(i) $x_P(t) = A\cos(2\,t)$ *(because there are only even derivatives on the left side.) It is also acceptable to guess* $x_P(t) = A\cos(2\,t) + B\sin(2\,t)$, *but it will turn out that* $B = 0$*).*

(ii) Since $\omega_0^2 = 5$, $x_H(t) = c_1\cos(\sqrt{5}\,t) + c_2\sin(\sqrt{5}\,t)$. *Thus the general solution is*

$$x(t) = x_P(t) + x_H(t) = A\cos(2\,t) + c_1\cos(\sqrt{5}\,t) + c_2\sin(\sqrt{5}\,t)$$
$$= A\cos(2\,t) + C\cos(\sqrt{5}\,t - \alpha).$$

(iii) Because $\sqrt{5}$ *is close to 2, solutions to this DE will exhibit beating.*

2b) $x''(t) + 4\,x(t) = 3\cos(2\,t)$

(8 points)

(i) $x_P(t) = t\,(A\cos(2\,t) + B\sin(2\,t))$ *because* $\omega = \omega_0 = 2$. *(If you remember how this computation actually worked out, then you might remember that the actual particular solution is of the form* $t\,(B\sin(2\,t))$.

The more precise explanation for this undetermined coefficient form is that the characteristic polynomial for the homogeneous DE is

$$p(r) = r^2 + 4 = (r + 2\,i)^1\,(r - 2\,i)^1$$

so the right side function solves the homogeneous DE, and corresponds to the complex roots $\pm 2\,i$. *Since the powers of the corresponding factors in* $p(r)$ *are 1, we multiply the guess we would've made if* $\pm 2\,i$ *were not roots of the characteristic polynomial by* t^1.

(ii) Since $\omega_0^2 = 4$, $x_H(t) = c_1\cos(2\,t) + c_2\sin(2\,t)$. *Thus the general solution is*

$$x(t) = x_P(t) + x_H(t) = t\,(A\cos(2\,t) + B\sin(2\,t)) + c_1\cos(2\,t) + c_2\sin(2\,t).$$

(iii) resonance, as shown by the particular solution which has an "amplitude" that is growing linearly in t.

2c) $x''(t) + 0.2\cdot x'(t) + 4.01\,x(t) = 3\cos(2\,t)$.

(9 points)

(i) $x_P(t) = A\cos(2\,t) + B\sin(2\,t)$.

(ii) $p(r) = r^2 + .2\,r + 4.01 = (r + .1)^2 + 4$ has roots

$$
\begin{aligned}
(r + .1)^2 + 4 &= 0 \\
(r + .1)^2 &= -4 \\
r + .1 &= \pm 2\,i \\
r &= -.1 \pm 2\,i
\end{aligned}
$$

so

$$x_H(t) = c_1 e^{-.1\,t}\cos(2\,t) + c_2\, e^{-.2\,t}\sin(2\,t)$$

and

$$x(t) = x_P + x_H = x_{sp} + x_{tr} = C_1 \cos\left(2\,t - \alpha_1\right) + C_2\, e^{-.1\,t}\cos\left(2\,t - \alpha_2\right).$$

Since $\omega = 2$ is close to the (undamped) $\omega_0 = \sqrt{4.01}$ the amplitude C_1 of the steady periodic solution will not be small, which is related to <u>practical resonance</u>.

3) Use Chapter 5 techniques to solve the initial value problem

$$x''(t) + 9\,x(t) = 10\cos(2\,t)$$
$$x(0) = 2$$
$$x'(0) = 1$$

(20 points)

$x_H(t) = c_1 \cos(3\,t) + c_2 \sin(3\,t)$ *since* $\omega_0 = 3$. *(You could also get here using the characteristic polynomial, but you should recognize this by now.)*

$x_P(t) = A\cos(2\,t) \Rightarrow L(x_P) = -4\,A\cos(2\,t) + 9\,A\cos(2\,t) = 5\,A\cos(2\,t)$, *which will equal* $10\cos(2\,t)$ *when* $A = 2$. *So we may take* $x_P(t) = 2\cos(2\,t)$.

Thus the general solution is given by

$$x(t) = x_P(t) + x_H(t) = 2\cos(2\,t) + c_1\cos(3\,t) + c_2\sin(3\,t).$$

For the IVP:

$$x(0) = 2 = 2 + c_1 \Rightarrow c_1 = 0\,.$$
$$x'(0) = 1 = 3\,c_2 \Rightarrow c_2 = \frac{1}{3}.$$

Thus

$$x(t) = 2\cos(2\,t) + \frac{1}{3}\sin(3\,t)\,.$$

4) Re-solve the initial value problem in (3) using Laplace transform.

$$x''(t) + 9x(t) = 10\cos(2t)$$
$$x(0) = 2$$
$$x'(0) = 1$$

(20 points)

$$s^2 X(s) - s\cdot 2 - 1 + 9(X(s)) = \frac{10s}{s^2+4}.$$

$$X(s)\left(s^2+9\right) = \frac{10s}{s^2+4} + 2s + 1$$

$$X(s) = \frac{10s}{\left(s^2+4\right)\left(s^2+9\right)} + \frac{2s}{s^2+9} + \frac{1}{s^2+9}.$$

$$= 10s\left(\frac{1}{5}\right)\left(\frac{1}{s^2+4} - \frac{1}{s^2+9}\right) + \frac{2s}{s^2+9} + \frac{1}{s^2+9}.$$

$$= \frac{2s}{s^2+4} - \frac{2s}{s^2+9} + \frac{2s}{s^2+9} + \frac{1}{s^2+9}$$

$$= \frac{2s}{s^2+4} + \frac{1}{3}\frac{3}{s^2+9}.$$

Thus

$$x(t) = 2\cos(2t) + \frac{1}{3}\sin(3t).$$

5a) Set up the partial fractions decomposition for

$$F(s) = \frac{4\,s + 10}{\left(s^2 + 4\,s + 13\right)(s+2)^2\left(s^2+16\right)^2}\,.$$

Do not try to find the actual values of the partial fraction coefficients.

(7 points)

$$F(s) = \frac{A\,s + B}{s^2 + 4\,s + 13} + \frac{C}{s+2} + \frac{\mathrm{D}}{(s+2)^2} + \frac{E\,s + F}{\left(s^2+16\right)} + \frac{G\,s + H}{\left(s^2+16\right)^2}\,.$$

5b) Use the Laplace transform table to find the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$ for the $F(s)$ is part (a), in terms of the (unknown) partial fraction coefficients.

(8 points)

The only term we need to adjust to use the table, is the first one - in which we have to "complete the linear" in the numerator, after completing the square in the denominator:

$$\frac{A\,s + B}{s^2 + 4\,s + 13} = \frac{A\,s + B}{(s+2)^2 + 9} = \frac{A(s+2) + (B - 2\,A)}{(s+2)^2 + 9}\,.$$

So

$$F(s) = \frac{A(s+2)}{(s+2)^2 + 9} + \frac{(B-2\,A)}{3}\,\frac{3}{(s+2)^2+9} + \frac{C}{s+2} + \frac{\mathrm{D}}{(s+2)^2} + \frac{E\,s+F}{\left(s^2+16\right)}$$
$$+ \frac{G\,s + H}{\left(s^2+16\right)^2}$$

and

$$f(t) = A\,e^{-2\,t}\cos(3\,t) + \frac{(B-2\,A)}{3}e^{-2\,t}\sin(3\,t) + C\,e^{-2\,t} + \mathrm{D}\,t\,e^{-2\,t}$$
$$+ E\cos(4\,t) + \frac{F}{4}\sin(4\,t) + \frac{G\,t}{8}\sin(2\,t) + \frac{H}{128}\left(\sin(4\,t) - 4\,t\cos(4\,t)\right).$$

For fun:

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>
> F := s -> (4*s + 10)/((s^2 + 4*s + 13)*(s + 2)^3*(s^2 + 16)^2):
  convert(F(s), parfrac, s);
```

$$\frac{1}{1800\,(s+2)^3} + \frac{1}{5688225}\,\frac{-7184 - 3214\,s}{s^2 + 4\,s + 13} + \frac{319}{810000\,(s+2)} + \frac{1}{53000}\,\frac{-414 + 137\,s}{(s^2+16)^2}$$
$$+ \frac{1}{28090000}\,\frac{-43338 + 4809\,s}{s^2 + 16} + \frac{1}{750\,(s+2)^2} \qquad \textbf{(1)}$$

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> with(inttrans):
  invlaplace(F(s), s, t);
```

$$\frac{3}{224720000}\cos(4\,t)\,(12824 + 18285\,t) + \frac{1}{2275290000}\left(896071 - 1285600\cos(3\,t) - 100800\sin(3\,t)\right.$$
$$\left. + 632025\,t^2 + 3033720\,t\right)\mathrm{e}^{-2\,t} + \frac{1}{898880000}\sin(4\,t)\,(-401559 + 290440\,t) \qquad \textbf{(2)}$$

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Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\dfrac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\dfrac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\dfrac{1}{2k^3}(\sin kt-kt\cos kt)$	$\dfrac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\dfrac{t}{2k}\sin kt$	$\dfrac{s}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^nF^{(n)}(s)$	$\dfrac{1}{2k}(\sin kt+kt\cos kt)$	$\dfrac{s^2}{(s^2+k^2)^2}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t)$, period p	$\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\dfrac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\dfrac{1}{s}\tanh\dfrac{as}{2}$
t	$\dfrac{1}{s^2}$	$\left[\!\!\left[\dfrac{t}{a}\right]\!\!\right]$ (staircase)	$\dfrac{e^{-as}}{s(1-e^{-as})}$
t^n	$\dfrac{n!}{s^{n+1}}$		
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$		
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$		