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Student I.D. _____

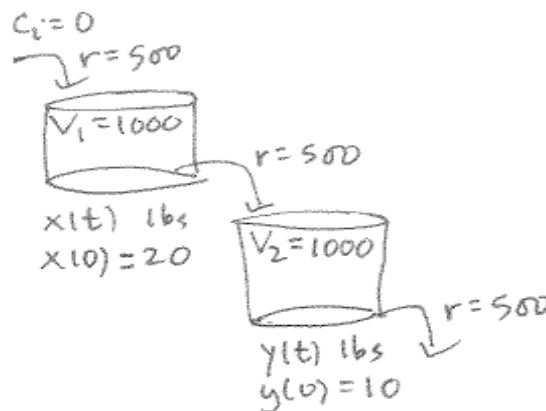
Math 2250-4
Exam #1 SOLUTIONS
February 16, 2012

Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not a "graphing calculator" i.e. not one which is capable of integration, taking derivatives, or matrix algebra. **In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. **Good Luck!**

1) Consider a cascade of two brine tanks, in which water flows into the first tank then from the first tank to the second one, and then out of the second tank. The water flowing into the first tank is pure, i.e. it contains no salt. All the flow rates are 500 gallons per hour, so that the water volumes $V_1 = 1000$ g, $V_2 = 1000$ g in each tank remain constant. Let $x(t)$ be the pounds of salt in tank 1, and let $y(t)$ be the pounds of salt in tank 2. Assume that at the moment the cascade is set into motion, there are 20 lbs of salt in the first tank, and 10 lbs in the second tank.

1a) Make a sketch of the configuration as described above. Include all of the information.

(5 points)



1b) Use your modeling ability to set up the initial value problem for $x(t)$. Then solve this differential equation (or identify its type), to deduce that

$$x(t) = 20 e^{-0.5 t}.$$

(10 points)

Solution:

$$\begin{aligned} x'(t) &= r_i c_i - r_o c_o \\ &= 500(0) - 500 \frac{x}{1000} = -0.5 x. \end{aligned}$$

So the IVP is

$$\begin{aligned} x'(t) &= -0.5 x \\ x(0) &= 20. \end{aligned}$$

But $x'(t) = -k x(t)$ is exponential decay, with solution $x(t) = x_0 e^{-k t}$, so in our case we deduce

$$x(t) = 20 e^{-.5 t}.$$

1c) Using the result of 1b) and your input-output modeling ability, show that $y(t)$ solves the initial value problem

$$\begin{aligned} y'(t) + 0.5 y(t) &= 10 e^{-0.5 t} \\ y(0) &= 10. \end{aligned}$$

(5 points)

Solution:

$$\begin{aligned} y'(t) &= r_i c_i - r_o c_o \\ y'(t) &= 500 \frac{x}{1000} - 500 \frac{y}{1000} \\ y'(t) &= .5 x - .5 y = 10 e^{-.5 t} - .5 y \\ y'(t) + 0.5 y(t) &= 10 e^{-.5 t} . \\ y(0) &= 10 \text{ (as given).} \end{aligned}$$

1d) Solve the initial value problem in 1c).

(10 points)

$$\begin{aligned} y'(t) + 0.5 y(t) &= 10 e^{-0.5 t} \\ e^{.5 t} (y'(t) + 0.5 y(t)) &= e^{.5 t} 10 e^{-0.5 t} = 10 . \\ \frac{d}{dt} (e^{.5 t} y(t)) &= 10 \\ e^{.5 t} y(t) &= \int 10 dt = 10 t + C . \end{aligned}$$

At $t = 0$, $y = 10$, so

$$\begin{aligned} 10 &= C . \\ e^{.5 t} y(t) &= 10 t + 10 = 10(t + 1) . \\ y(t) &= 10(t + 1)e^{-.5 t} . \end{aligned}$$

1e) At what time will the salt amount $y(t)$ in tank 2 be at a maximum?

(5 points)

when $y'(t) = 0$:

$$\begin{aligned} y'(t) &= 10[1e^{-.5 t} + (t + 1)(-.5)e^{-.5 t}] . \\ 0 &= e^{-.5 t}(1 - .5(t + 1)) \Rightarrow 1 - .5(t + 1) = 0 \Rightarrow .5(t + 1) = 1 \Rightarrow t = 1 . \end{aligned}$$

So $y(t)$ will be at a maximum at $t = 1$ hour.

Alternately, you could use the DE as a shortcut for figuring out the critical time, when $y'(t) = 0$:

$$\begin{aligned} y'(t) + 0.5 y(t) &= 10 e^{-0.5 t} \\ 0 + .5(10)(t + 1)e^{-.5 t} &= 10 e^{-0.5 t} \Rightarrow \\ 5(t + 1) &= 10 \Rightarrow t = 1 \text{ hour.} \end{aligned}$$

You weren't asked for it, but the maximum value is $y(1) = 20e^{-.5}$, pounds, a little over 12 pounds:

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> Digits := 5 :
  20*exp(-.5);
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12.131

(1)

2) A certain population $P(t)$ of animals satisfies a differential equation,

$$P'(t) = -2P^2 + 10P - 8.$$

2a) Find the equilibrium populations for this differential equation. Then make a phase diagram and determine the stability of the equilibrium solutions.

(8 points)

$$P'(t) = -2(P^2 - 5P + 4) = -2(P - 4)(P - 1)$$

so the equilibrium solutions are $P \equiv 4$, $P \equiv 1$.

For $P > 4$, we have $P'(t) < 0$; For $1 < P < 4$ we have $P'(t) > 0$. For $P < 1$ we have $P'(t) < 0$.
So the phase diagram is

$$\leftarrow \leftarrow \leftarrow 1 \rightarrow \rightarrow \rightarrow 4 \leftarrow \leftarrow \leftarrow \leftarrow .$$

Thus $P \equiv 1$ is unstable and $P \equiv 4$ is asymptotically stable.

2b) Into which category of the population models that we've discussed does this differential equation fall?
(3 points)

This is logistic with harvesting: If the DE just had the first two terms

$$P'(t) = -2P^2 + 10P = 2P(5 - P)$$

it would be logistic, with carrying capacity $P = 5$. In the actual differential equation

$$P'(t) = -2P^2 + 10P - 8$$

the term "-8" represents constant rate harvesting.

2c) If the initial population in this model is $P_0 = 3$, what will the limiting population to be, based upon your work in (2a)?

(3 points)

Since $1 < P_0 < 4$ the phase diagram implies that $\lim_{t \rightarrow \infty} P(t) = 4$.

2d) Solve the initial value problem for this population model:

$$P'(t) = -2P^2 + 10P - 8$$

$$P(0) = 3$$

You may use the partial fractions identity

$$\frac{1}{(P-a)(P-b)} = \frac{1}{b-a} \left(\frac{1}{P-b} - \frac{1}{P-a} \right)$$

(12 points)

$$P'(t) = -2(P-4)(P-1)$$

$$\frac{dP}{(P-4)(P-1)} = -2 dt$$

$$\frac{1}{3} \left(\frac{1}{P-4} - \frac{1}{P-1} \right) dP = -2 dt$$

$$\int \left(\frac{1}{P-4} - \frac{1}{P-1} \right) dP = \int -6 dt$$

$$\ln \left| \frac{P-4}{P-1} \right| = -6t + C_1$$

$$\left| \frac{P-4}{P-1} \right| = e^{C_1} e^{-6t}$$

$$\frac{P-4}{P-1} = C e^{-6t}$$

$$P_0 = 3 \Rightarrow \frac{-1}{2} = C \Rightarrow$$

$$\frac{P-4}{P-1} = -\frac{1}{2} e^{-6t}$$

$$P-4 = -\frac{1}{2} e^{-6t} (P-1)$$

$$P-4 = -\frac{1}{2} e^{-6t} P + \frac{1}{2} e^{-6t}$$

$$P \cdot \left(1 + \frac{1}{2} e^{-6t} \right) = 4 + \frac{1}{2} e^{-6t}$$

$$P(t) = \frac{4 + \frac{1}{2} e^{-6t}}{1 + \frac{1}{2} e^{-6t}}$$

$$\text{check: } P(0) = \frac{4 + \frac{1}{2}}{1 + \frac{1}{2}} = \frac{\frac{9}{2}}{\frac{3}{2}} = \frac{9}{3} = 3. \quad \heartsuit$$

2e) Explain why your predicted limiting population from (2b) is consistent with your actual solution from (2d).

(4 points)

$$\lim_{t \rightarrow \infty} \frac{4 + \frac{1}{2} e^{-6t}}{1 + \frac{1}{2} e^{-6t}} = \frac{4 + 0}{1 + 0} = 4 \text{ which is what we deduced in (2c).}$$

3a) Show that $y(x) = c_1 e^{2x} + c_2 e^{-3x}$ solves the differential equation

$$y'' + y' - 6y = 0.$$

(5 points)

We'll compute y, y', y'' in successive lines, multiply by the appropriate scalars, and then add up column by column to compute $y'' + y' - 6y$:

$$\begin{aligned} & -6[y = c_1 e^{2x} + c_2 e^{-3x}] \\ & + 1[y' = 2c_1 e^{2x} - 3c_2 e^{-3x}] \\ & + 1[y'' = 4c_1 e^{2x} + 9c_2 e^{-3x}] \\ & -6y + y' + y'' = c_1 e^{2x}(-6 + 2 + 4) \\ & \quad + c_2 e^{-3x}(-6 - 3 + 9) \\ & -6y + y' + y'' = 0 + 0 = 0 \end{aligned}$$

3b) Find constants c_1, c_2 for $y(x)$ as above, to find a solution to the initial value problem

$$y'' + y' - 6y = 0.$$

$$y(0) = 5$$

$$y'(0) = -5$$

(5 points)

$$\begin{aligned} y &= c_1 e^{2x} + c_2 e^{-3x} \\ y' &= 2c_1 e^{2x} - 3c_2 e^{-3x} \\ y(0) &= c_1 + c_2 = 5 \\ y'(0) &= 2c_1 - 3c_2 = -5. \end{aligned}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ -5 \end{bmatrix} = \frac{1}{-5} \begin{bmatrix} -3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -5 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} -10 \\ -15 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

So, $y(x) = 2e^{2x} + 3e^{-3x}$. (Check: $y(0) = 2 + 3 = 5$, $y'(0) = 4 - 9 = -5$ ♥)

4) Consider the matrix A defined by

$$A := \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 1 \\ -2 & 3 & -6 \end{bmatrix}$$

4a) Compute the determinant of A .

(4 points)

For example, we could expand down the first column:

$$|A| = 1 \begin{vmatrix} 1 & 1 \\ 3 & -6 \end{vmatrix} - 0 + (-2) \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} = -9 + 8 = -1.$$

4b) What does the value of the determinant you computed in (4a) tell you about whether or not the matrix A has an inverse? Explain, by quoting the general fact relating determinant values to matrix inverses.

(4 points)

Since $|A| \neq 0$, A^{-1} exists.

4c) Find the inverse matrix to A , using either of the two algorithms we've learned. Remember to show all work, as always. Hint: The correct inverse matrix has no fractions in it, which you could deduce from your work in the answer to part (4a)

(12 points)

Using row operations:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -2 & 3 & -6 & 0 & 0 & 1 \end{array} \right]$$

$2R_1 + R_3$ in R_3 :

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \end{array} \right]$$

$-R_2 + R_3$ in R_3 :

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 1 \end{array} \right]$$

$-R_3$ in R_3 :

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & -1 \end{array} \right]$$

$-R_3 + R_2$ in R_2 ; $-3R_3 + R_1$ in R_1 :

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 7 & -3 & 3 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & -1 \end{array} \right]$$

$R_2 + R_1$ in R_1 :

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 9 & -3 & 4 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & -1 \end{array} \right]$$

$$\text{Thus } A^{-1} = \begin{bmatrix} 9 & -3 & 4 \\ 2 & 0 & 1 \\ -2 & 1 & -1 \end{bmatrix}.$$

$$(\text{Check: } \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 1 \\ -2 & 3 & -6 \end{bmatrix} \begin{bmatrix} 9 & -3 & 4 \\ 2 & 0 & 1 \\ -2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \heartsuit)$$

Alternately, you could use the adjoint formula:

$$A := \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 1 \\ -2 & 3 & -6 \end{bmatrix}$$

$$\text{cof}(A) = \begin{bmatrix} \begin{vmatrix} 1 & 1 \\ 3 & -6 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ -2 & -6 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -2 & 3 \end{vmatrix} \\ -\begin{vmatrix} -1 & 3 \\ 3 & -6 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ -2 & -6 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix} \\ \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -9 & 2 & 2 \\ 3 & 0 & -1 \\ -4 & -1 & 1 \end{bmatrix}$$

$$\text{Adj}(A) = \text{cof}(A)^T = \begin{bmatrix} -9 & 3 & -4 \\ 2 & 0 & -1 \\ 2 & -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj}(A) = - \begin{bmatrix} -9 & 3 & -4 \\ 2 & 0 & -1 \\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -3 & 4 \\ 2 & 0 & 1 \\ -2 & 1 & -1 \end{bmatrix}.$$

4d) Use the inverse matrix from part (4c) to solve the system

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 1 \\ -2 & 3 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -11 \end{bmatrix}$$

(5 points)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 1 \\ -2 & 3 & -6 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 0 \\ -11 \end{bmatrix} = \begin{bmatrix} 9 & -3 & 4 \\ 2 & 0 & 1 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ -11 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

(Check: $\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 1 \\ -2 & 3 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -11 \end{bmatrix}$ ♥.)