Math 2250-4
Mon Apr 9
7.1-7.2 Systems of differential equations - to model multi-component systems. Reduction of higher order IVPs to first order system IVPs. Theory and computation for first order IVPs.

• Recall from Friday's notes, that IVPs for higher order differential equations or systems of differential equations are equivalent to IVPs for first order systems of DEs.

• We were working an example of this equivalence, Exercise 5 which I've copied into today's notes as Exercise 1. I've also changed the initial conditions to make the constants work out more nicely than they did Friday. You'll be using this (new) IVP in your homework this week.

$$x'' + 2x' + 10x = 0$$
$$x(0) = 4$$
$$x'(0) = -4 .$$

On Friday we worked out that

$$p(r) = r^2 + 2r + 10 = (r+1)^2 + 9$$

which has roots $r = -1 \pm 3i$ so that

$$x_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) .$$

For the new and improved initial conditions you can immediately check that the solution is

$$x(t) = 4e^{-t}\cos(3t).$$

Exercise 1)
1a) Convert this single second order IVP into an equivalent first order system IVP for $x(t)$ and $v(t) := x'(t)$. Write this system in matrix-vector form.
We did this on Friday. Check that this is what we got:

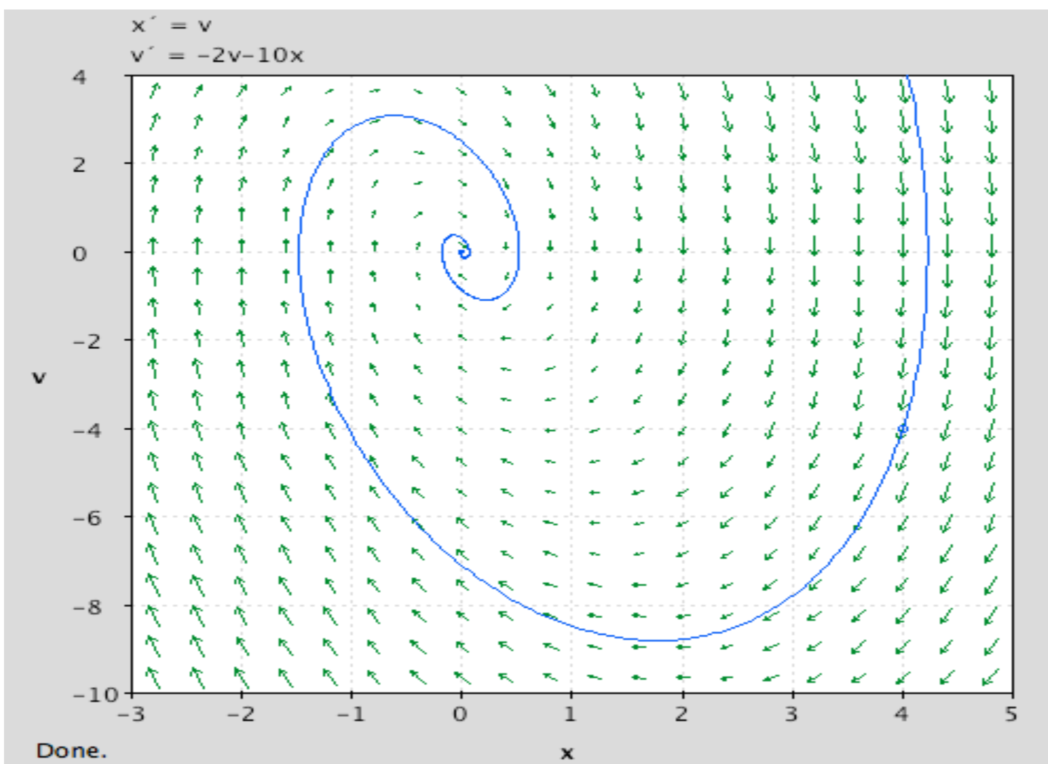
$$\begin{bmatrix} x'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

1b) Use the second order IVP solution to deduce a solution to the first order IVP.

1c) How does the Chapter 5 "characteristic polynomial" for the second order homogeneous DE compare with the Chapter 6 (eigenvalue) "characteristic polynomial" for the first order system matrix in 1a?

1d) Is your analytic solution $[x(t), v(t)]$ in 1b consistent with the parametric curve shown below? (This screenshot was generated with "pplane", the sister program to "dfield" that we used in Chapters 1-2.)



- After completing this exercise, finish pages 5-6 of Friday's notes.

Here are the Theorems for first order systems of differential equations. They are analogous to the ones we discussed for first order scalar DE IVPs back in Chapter 1.

Theorem 1   For the IVP

$$\underline{\mathbf{x}}'(t) = \underline{\mathbf{F}}(t, \underline{\mathbf{x}}(t))$$
$$\underline{\mathbf{x}}(t_0) = \underline{\mathbf{x}}_0$$

If $\underline{\mathbf{F}}(t, \underline{\mathbf{x}})$ is continuous in the t-variable and differentiable in its $\underline{\mathbf{x}}$ variable, then there exists a unique solution to the IVP, at least on some (possibly short) time interval $t_0 - \delta < t < t_0 + \delta$.

Theorem 2   For the special case of the first order linear system of differential equations IVP

$$\underline{\mathbf{x}}'(t) = A(t)\underline{\mathbf{x}}(t) + \underline{\mathbf{f}}(t)$$
$$\underline{\mathbf{x}}(t_0) = \underline{\mathbf{x}}_0$$

If the matrix $A(t)$ and the vector function $\underline{\mathbf{f}}(t)$ are continuous on an open interval I containing t_0 then a solution $\underline{\mathbf{x}}(t)$ exists and is unique, on the entire interval.

Remark: The solutions to these systems of DE's may be approximated numerically using vectorized versions of Euler's method and the Runge Kutta method. The ideas are exactly the same as they were for scalar equations, except that they now use vectors. This is how commerical numerical DE solvers work. For example, with time-step h the Euler loop would increment as follows:

$$t_j = t_0 + h\,j$$
$$\underline{\mathbf{x}}_{j+1} = \underline{\mathbf{x}}_j + h\,\underline{\mathbf{F}}(t_j, \underline{\mathbf{x}}_j).$$

Remark: These theorems are the true explanation for why the n^{th}-order linear DE IVPs in Chapter 5 always have unique solutions. In fact, when software finds numerical approximations for solutions to higher order DE IVPs that can't be found by the techniques of Chapter 5 or other mathematical formulas, it works by converting these IVPs to the equivalent first order system IVPs, and uses the algorithms described above to approximate the solutions.

Theorem 3) Vector space theory for first order systems of linear DEs (Notice the familiar themes...we can completely understand these facts if we take the intuitively reasonable existence-uniqueness Theorem 2 as fact.)

3.1) For vector functions $\underline{\mathbf{x}}(t)$ differentiable on an interval, the operator

$$L(\underline{\mathbf{x}}(t)) := \underline{\mathbf{x}}'(t) - A(t)\underline{x}(t)$$

is linear, i.e.

$$L(\underline{\mathbf{x}}(t) + \underline{\mathbf{z}}(t)) = L(\underline{\mathbf{x}}(t)) + L(\underline{\mathbf{z}}(t))$$
$$L(c\,\underline{\mathbf{x}}(t)) = c\,L(\underline{\mathbf{x}}(t)).$$

check!

3.2) Thus, by the fundamental theorem for linear transformations, the general solution to the non-homogeneous linear problem

$$\underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{f}}(t)$$

$\forall\, t \in I$ is

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}}_p(t) + \underline{\boldsymbol{x}}_H(t)$$

where $\underline{\boldsymbol{x}}_p(t)$ is any single particular solution and $\underline{\boldsymbol{x}}_H(t)$ is the general solution to the homogeneous problem

$$\underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{0}}$$

(which we also frequently rewrite as $\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$.)

3.3) For $A(t)_{n \times n}$ and $\underline{\boldsymbol{x}}(t) \in \mathbb{R}^n$ the solution space on the $t-$interval I to the homogeneous problem

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

is n-dimensional. Here's why:

- Let $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), \ldots \underline{\boldsymbol{X}}_n(t)$ be any n solutions to the homogeneous problem chosen so that the Wronskian matrix at $t_0 \in I$

$$[W(\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \ldots, \underline{\boldsymbol{X}}_n](t_0) := [\underline{\boldsymbol{X}}_1(t_0) | \underline{\boldsymbol{X}}_2(t_0) | \cdots | \underline{\boldsymbol{X}}_n(t_0)]$$

is invertible. (By the existence theorem we can choose solutions for any collection of initial vectors - so for example, in theory we could pick the matrix above to actually equal the identity matrix. In practice we'll be happy with any invertible matrix.)

- Then for any $\underline{\boldsymbol{b}} \in \mathbb{R}^n$ the IVP

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{b}}$$

has solution $\underline{\boldsymbol{x}}(t) = c_1\underline{\boldsymbol{X}}_1(t) + c_2\underline{\boldsymbol{X}}_2(t) + \ldots + c_n\underline{\boldsymbol{X}}_n(t)$ where the linear combination coefficients are the solution to the Wronskian matrix equation

$$\left[\begin{array}{c|c|c|c} & & & \\ \underline{\boldsymbol{X}}_1(t_0) & \underline{\boldsymbol{X}}_2(t_0) & \cdots & \underline{\boldsymbol{X}}_n(t_0) \\ & & & \end{array}\right]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Thus, because the Wronskian matrix at t_0 is invertible, every IVP can be solved with a linear combination of $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), \ldots \underline{\boldsymbol{X}}_n(t)$, and since each IVP has only one solution, $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), \ldots \underline{\boldsymbol{X}}_n(t)$ span the solution space. The same matrix equation shows that the only linear combination that yields the zero function (which has initial vector $\underline{\boldsymbol{b}} = \underline{\boldsymbol{0}}$) is the one with $\underline{\boldsymbol{c}} = \underline{\boldsymbol{0}}$. Thus $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), \ldots \underline{\boldsymbol{X}}_n(t)$ are also linearly independent. Therefore they are a basis for the solution space, and their number n is the dimension of the solution space.

7.3 Eigenvector method for finding the general solution to the homogeneous constant matrix first order system of differential equations

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

Here's how: We look for a basis of solutions $\underline{\boldsymbol{x}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$, where $\underline{\boldsymbol{v}}$ is a constant vector. Substituting this form of potential solution into the system of DE's above yields the equation

$$\lambda\, e^{\lambda t}\underline{\boldsymbol{v}} = A\, e^{\lambda t}\underline{\boldsymbol{v}} = e^{\lambda t} A\,\underline{\boldsymbol{v}}\,.$$

Dividing both sides of this equation by the scalar function $e^{\lambda t}$ gives the condition

$$\lambda\,\underline{\boldsymbol{v}} = A\,\underline{\boldsymbol{v}}\,.$$

- We get a solution every time $\underline{\boldsymbol{v}}$ is an eigenvector of A with eigenvalue λ !
- If A is diagonalizable then there is an $\mathbb{R}^n$ basis of eigenvectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ and solutions

$$\underline{\boldsymbol{X}}_1(t) = e^{\lambda_1 t}\,\underline{\boldsymbol{v}}_1,\ \underline{\boldsymbol{X}}_2(t) = e^{\lambda_2 t}\,\underline{\boldsymbol{v}}_2\,,\ \ldots,\ \underline{\boldsymbol{X}}_n(t) = e^{\lambda_n t}\,\underline{\boldsymbol{v}}_n$$

which are a basis for the solution space on the interval $I = \mathbb{R}$, because the Wronskian matrix at $t = 0$ is the invertible diagonalizing matrix

$$P = \left[\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \cdots | \underline{\boldsymbol{v}}_n\right]$$

that we considered in Chapter 6.

- If A has complex number eigenvalues and eigenvectors it may still be diagonalizable over $\mathbb{C}^n$, and we will still be able to extract a basis of real vector function solutions. If A is not diagonalizable over $\mathbb{R}^n$ or over $\mathbb{C}^n$ the situation is more complicated.

Exercise 2a) Use the method above to find the general homogeneous solution to

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

b) Solve the IVP with

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$