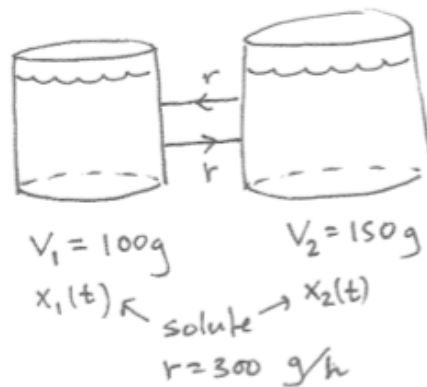Math 2250-4
Fri Apr 6
7.1 Systems of differential equations - to model multi-component systems.

<u>Examples:</u>

1) input-output models...related to "multi-component modeling", "compartmental analysis"

Here's a simple 2-tank problem to illustrate the ideas:

$V_1 = 100g$ $V_2 = 150g$

$x_1(t) \xleftarrow{} \text{solute} \xrightarrow{} x_2(t)$

$r = 300 \ g/h$

<u>Exercise 1)</u> Find differential equations for solute amounts $x_1(t), x_2(t)$ above, using input-output modeling. Assume solute concentration is uniform in each tank. If $x_1(0) = b_1, x_2(0) = b_2$, write down the initial value problem that you expect would have a unique solution.

answer (in matrix-vector form):

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

2) Coupled masses and springs (or RLC circuit loops) lead to systems of second order differential equations. For example,
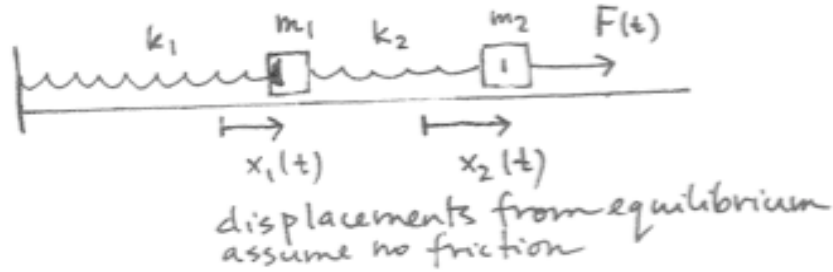
Exercise 2) Use Newton's second law to derive a system of two second order differential equations for $x_1(t), x_2(t)$, the displacements of the respective masses from the equilibrium configuration. What initial value problem do you expect yields unique solutions in this case?

Exercise 3) Let $m_1 = 2, m_2 = 1; k_1 = 4, k_2 = 2; F(t) = 40 \sin(3\,t)$. Show that in matrix-vector form the system above is equivalent to

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 40 \sin(3\,t) \end{bmatrix}.$$

3) It is always possible to convert a system of higher order differential equations into an equivalent (but larger) system of first order differential equations. This turns out to be important conceptually but also for numerical (computational) methods for solving systems of differential equations. The conversions relies on introducing auxillary functions for intermediate derivatives. For example,

Exercise 4) Consider the IVP from Exercise 3:
$$x_1'' = -3 x_1 + x_2$$
$$x_2'' = 2 x_1 - 2 x_2 + 40 \sin(3 t)$$
$$x_1(0) = b_1, x_1'(0) = b_2$$
$$x_2(0) = c_1, x_2'(0) = c_2 .$$

4a) Show that if $x_1(t), x_2(t)$ solve the IVP above, and if we define
$$v_1(t) := x_1'(t)$$
$$v_2(t) := x_2'(t)$$
then $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the first order system IVP
$$x_1' = v_1$$
$$x_2' = v_2$$
$$v_1' = -3 x_1 + x_2$$
$$v_2' = 2 x_1 - 2 x_2 + 40 \sin(3 t)$$
$$x_1(0) = b_1$$
$$v_1(0) = b_2$$
$$x_2(0) = c_1$$
$$v_2(0) = c_2 .$$

4b) Conversely, show that if $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the first order IVP, then $x_1(t), x_2(t)$ solve the original second order IVP.

4) A special case of the discussion on the previous page is if you have a single higher order DE for $x(t)$; it is equivalent to a first order system for
$$x_1(t), y_1(t) := x'(t), y_2(t) := x''(t), \dots, y_{n-1}(t) := x_1^{(n-1)}(t).$$
For example, consider this second order underdamped IVP for $x(t)$:
$$x'' + 2x' + 10x = 0$$
$$x(0) = 4$$
$$x'(0) = 0 \ .$$

Exercise 5)

5a) Convert this single second order IVP into an equivalent first order system IVP for $x(t)$ and $v(t) := x'(t)$.

5b) Solve the second order IVP in order to deduce a solution to the first order IVP. Use Chapter 5 methods even though you love Laplace transform more.

5c) How does the Chapter 5 "characteristic polynomial" in 5b compare with the Chapter 6 (eigenvalue) "characteristic polynomial" for the first order system matrix in 5a? hmmm.

5d) Is your analytic solution $[x(t), v(t)]$ in 5b consistent with the parametric curve shown below? (This screenshot was generated with "pplane", the sister program to "dfield" that we used in Chapters 1-2.)
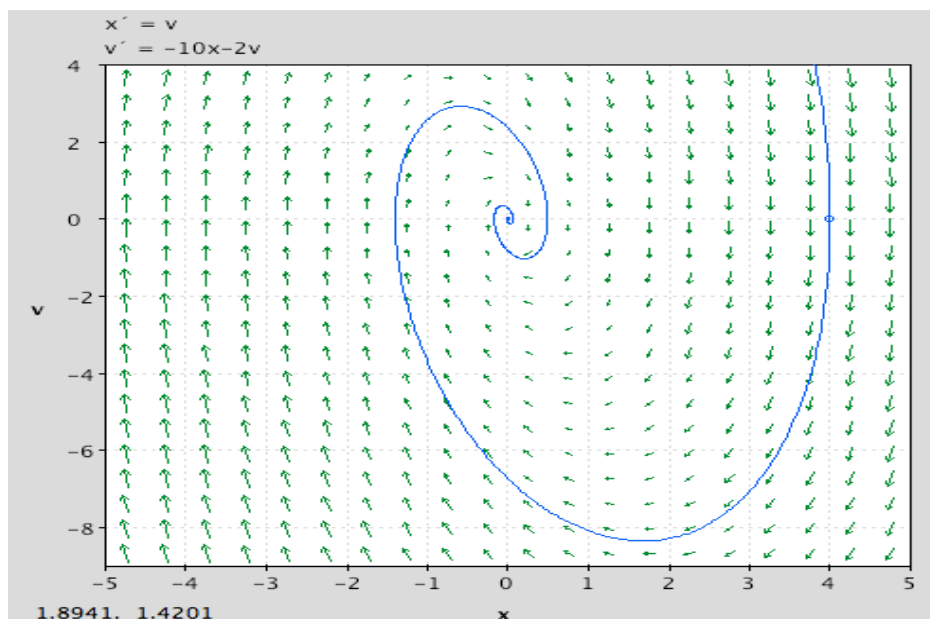
<u>Geometric interpretation of first order systems of differential equations.</u> (And how numerical differential equation solvers like pplane work.)

Consider the general initial value problem for a first order system of differential equations:
$$\underline{x}'(t) = \underline{F}(t, \underline{x}(t))$$
$$\underline{x}(t_0) = \underline{x}_0$$

• From either a multivariable calculus course, or from physics, recall the geometric/physical interpretation of $\underline{x}'(t)$ as the tangent/velocity vector to the parametric curve of points with position vector $\underline{x}(t)$, as t varies. This picture should remind you of the discussion, but ask questions if this is new to you:
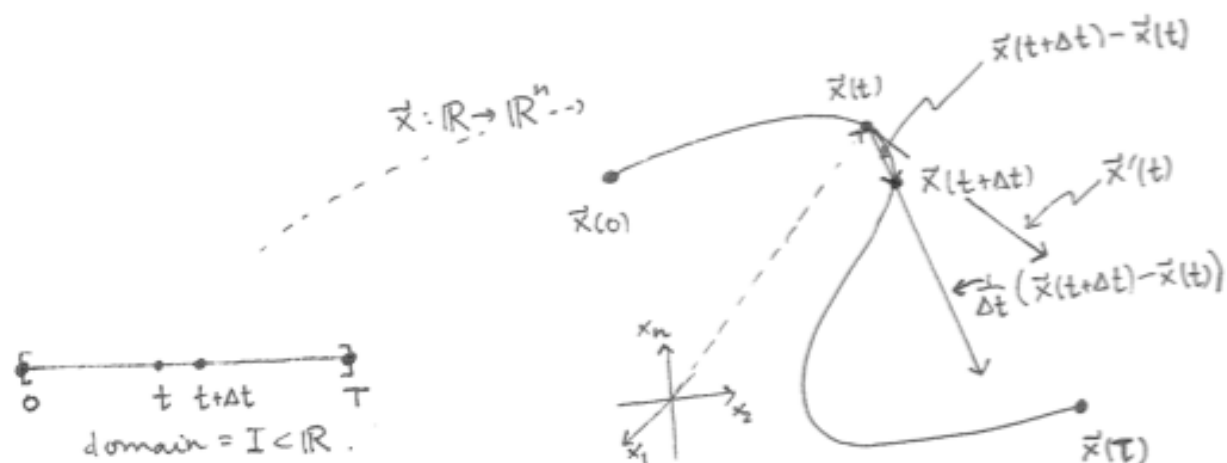
Analytically, the reason that the vector of derivatives $\underline{x}'(t)$ computed component by component is actually a limit of scaled secant vectors (and therefore a tangent/velocity vector) is:

$$\underline{x}'(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\begin{bmatrix} x_1(t + \Delta t) \\ x_2(t + \Delta t) \\ \vdots \\ x_n(t + \Delta t) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \right)$$

$$= \lim_{\Delta t \to 0} \begin{bmatrix} \frac{1}{\Delta t}\left(x_1(t + \Delta t) - x_1(t) \right) \\ \frac{1}{\Delta t}\left(x_2(t + \Delta t) - x_2(t) \right) \\ \vdots \\ \frac{1}{\Delta t}\left(x_n(t + \Delta t) - x_n(t) \right) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} ,$$

provided each component function is differentiable. Therefore, the reason you expect a unique solution to the IVP for a first order system is that you know where you start ($\underline{x}(t_0) = \underline{x}_0$), and you know your "velocity" vector (depending on time and current location) $\Rightarrow$ you <u>expect a unique solution!</u> (Plus, you could use something like a vector version of Euler's method or the Runge-Kutta method to approximate it!)

<u>Exercise 6)</u> Return to the page 1 tank example

$$x_1{}'(t) = -3\,x_1 + 2\,x_2$$
$$x_2{}'(t) = 3\,x_1 - 2\,x_2$$
$$x_1(0) = 0$$
$$x_2(0) = 10$$

<u>6a)</u> Interpret the parametric solution curve $[x_1(t), x_2(t)]^T$ to this IVP, as indicated in the pplane screen shot below. Notice how it follows the "velocity" vector field (which is time-independent), and how the "particle motion" location $[x_1(t), x_2(t)]^T$ is actually the vector of solute amounts in each tank. If your system involved ten coupled tanks rather than two, then this "particle" is moving around in $\mathbb{R}^{10}$.

<u>6b)</u> What are the apparent limiting solute amounts in each tank?

<u>6c)</u> How could your smart-alec younger sibling have told you the answer to <u>6b</u> without considering any differential equations at all?