Math 2250-4
Wed Apr 4
6.2 Eigenvalues and eigenvectors for square matrices; diagonalizability of matrices.

Recall from yesterday,

Definition: If $A_{n \times n}$ and if $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$ for some scalar λ and vector $\underline{\boldsymbol{v}} \neq \underline{\boldsymbol{0}}$ then $\underline{\boldsymbol{v}}$ is called an eigenvector of A, and λ is called the eigenvalue of $\underline{\boldsymbol{v}}$ (and an eigenvalue of A).

- For a diagonal matrix, the standard basis vectors are eigenvectors, and the corresponding matrix entries along the diagonal are their eigenvalues. These diagonal entries are the only eigenvalues of the matrix.

- For general matrices, the eigenvector equation $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$ can be rewritten as

$$(A - \lambda I)\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}} .$$

Thus the only possible eigenvalues associated to a given matrix must be roots λ_j of the characteristic polynomial

$$p(\lambda) = det(A - \lambda I) .$$

Thus, the first step in finding eigenvectors for (a non-diagonal) A is actually to find the eigenvalues - by finding the roots λ_j of the characteristic polynomial.

- For each root λ_j there will be one or more independent eigenvectors. You can find a basis for the λ_j eigenspace E_{λ_j} by solving the homogeneous matrix equation

$$(A - \lambda_j I)\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$$

e.g. by reducing the augmented matrix, backsolving, and extracting a basis.

- If your matrix is diagonal, the general algorithm above just reproduces the entries along the diagonal as eigenvalues, and the corresponding standard basis vectors as eigenspace bases. (So, for a diagonal matrix, don't go through all that extra work!)

Exercise 1) Finish the last exercise from Tuesday. We were considering the matrix

$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} .$$

step 1) Yesterday we computed

$$|B - \lambda I| = \begin{bmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 2 & -2 & 3-\lambda \end{bmatrix} = -(\lambda-2)^2(\lambda-3) .$$

step 2) Find bases for the eigenspaces E_2 and E_3 . Talk about how you are able to use the fact that homogeneous solutions to matrix equations correspond to column dependencies, in order to read off eigenspace bases more quickly than by backsolving.... ask questions if this is not clear, because you'll be computing a lot of eigenvalues and eigenvectors in the next chapters and you'll save a lot of time if you get comfortable with this shortcut. (Of course, for most larger matrices one just uses technology for eigenvalue/eigenvector computation.)

geometric interpretation: Notice that you can construct a basis for $\mathbb{R}^3$ by combining your eigenspace bases above. Use this fact to describe the geometry of the transformation

$$T(\underline{\boldsymbol{x}}) = B\,\underline{\boldsymbol{x}}\,.$$

Your algebraic work above is related to the output below:

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> with(LinearAlgebra):
  B := Matrix(3, 3, [4, -2, 1, 2, 0, 1, 2, -2, 3]);
  Eigenvectors(B);
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$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & -\frac{1}{2} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad (1)$$

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In all of our examples so far, it turns out that by collecting bases from each eigenspace for the matrix $A_{n \times n}$, and putting them together, we get a basis for $\mathbb{R}^n$. This lets us understand the <u>geometry</u> of the transformation

$$T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$$

almost as well as if A is a diagonal matrix. Having such a basis of eigenvectors is also extremely useful for <u>algebraic</u> computations:

Use the $\mathbb{R}^3$ basis made of out eigenvectors of the matrix B in <u>Exercise 1</u>, and put them into the columns of a matrix we will call P. We could order the eigenvectors however we want, but we'll put the E_2 basis vectors in the first two columns, and the E_3 basis vector in the third column:

$$P := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix} .$$

Now do algebra (check these steps and discuss what's going on!)

$$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 4 & -4 & 3 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{array}\right] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} .$$

In other words,

$$B\,P = P\,\mathrm{D}\,,$$

where D is the diagonal matrix of eigenvalues (for the corresponding columns of eigenvectors in P).
Equivalently (multiply on the right by P^{-1})

$$B = P\,\mathrm{D}\,P^{-1} .$$

<u>Exercise 2)</u> Use the identity above to show how B^{100} can be computed with only two matrix multiplications!

<u>Definition:</u> Let $A_{n \times n}$. If there is an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ consisting of eigenvectors of A, then A is called <u>diagonalizable.</u> This is why:

Write $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$ (some of these λ_j may be the same, as in the previous example). Let P be the matrix

$$P = \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right].$$

Then, using the various ways of understanding matrix multiplication, we see

$$A\,P = A\left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] = \left[\lambda_1\underline{\boldsymbol{v}}_1 \middle| \lambda_2\underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \lambda_n\underline{\boldsymbol{v}}_n\right]$$

$$= \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

$$A\,P = P\,\mathrm{D}$$

$$A = P\,\mathrm{D}\,P^{-1}$$

$$P^{-1}A\,P = \mathrm{D}\,.$$

Unfortunately, not all matrices are diagonalizable:

<u>Exercise 3)</u> Show that

$$C := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is <u>not</u> diagonalizable.

Facts about diagonalizability (see text section 6.2 for complete discussion, with reasoning):

Let $A_{n \times n}$ have factored characteristic polynomial

$$p(\lambda) = (\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2} \ldots (\lambda - \lambda_m)^{k_m}$$

where like terms have been collected so that each λ_j is distinct (i.e different). Notice that

$$k_1 + k_2 + \ldots + k_m = n$$

because the degree of $p(\lambda)$ is n.

- Then $1 \le dim\left(E_{\lambda_j}\right) \le k_j$. If $dim\left(E_{\lambda_j}\right) < k_j$ then the λ_j eigenspace is called defective.
- The matrix A is diagonalizable if and only if each $dim\left(E_{\lambda_j}\right) = k_j$. In this case, one obtains an $\mathbb{R}^n$ eigenbasis simply by combining bases for each eigenspace into one collection of n vectors. (Later on, the same definitions and reasoning will apply to complex eigenvalues and eigenvectors, and a basis of $\mathbb{C}^n$.)

- In the special case that A has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ each eigenspace is forced to be 1-dimensional since $k_1 + k_2 + \ldots + k_n = n$ so each $k_j = 1$. Thus A is automatically diagonalizable as a special case of the second bullet point.

Exercise 4) How do the examples from today and yesterday compare with the general facts about diagonalizability?