

The study of eigenvalues and eigenvectors is a return to matrix linear algebra, and the concepts we discuss will help us study linear systems of differential equations, in Chapter 7. Such systems of DE's arise naturally in the contexts of

- coupled input-output models, with several components.
- coupled mass-spring or RLC circuit loops, with several components.

To introduce the idea of eigenvalues and eigenvectors we'll first think geometrically.

Example Consider the matrix transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Notice that for the standard basis vectors  $\mathbf{e}_1 = [1, 0]^T$ ,  $\mathbf{e}_2 = [0, 1]^T$

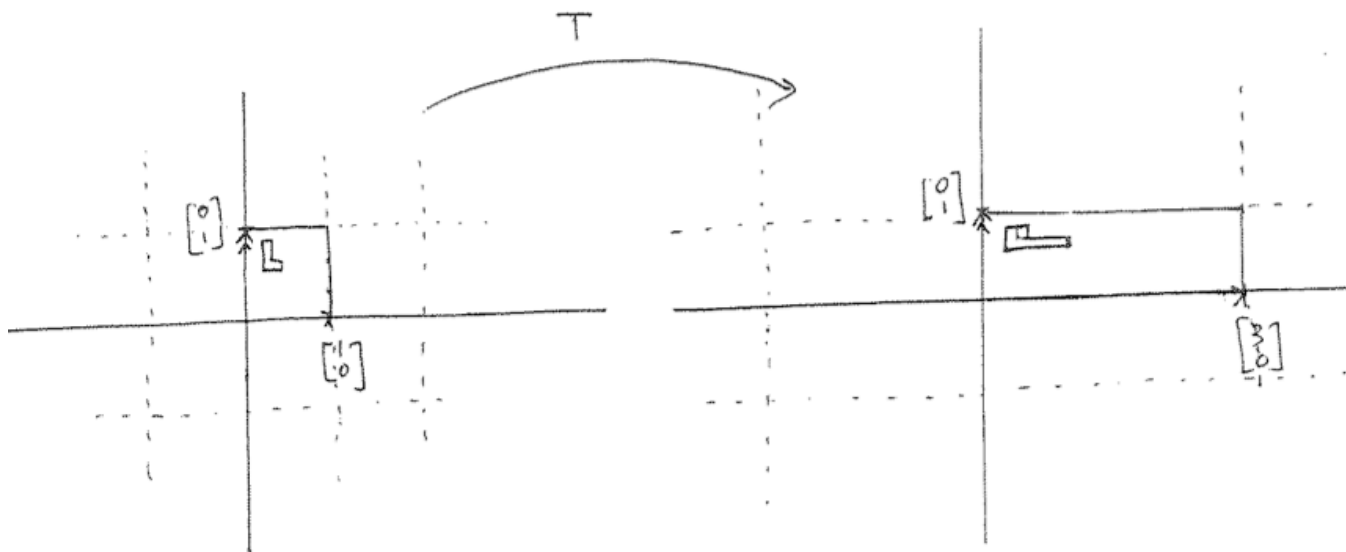
$$T(\mathbf{e}_1) = 3\mathbf{e}_1$$

$$T(\mathbf{e}_2) = \mathbf{e}_2.$$

The facts that  $T$  is linear and that it transforms  $\mathbf{e}_1, \mathbf{e}_2$  by scalar multiplying them, lets us understand the geometry of this transformation completely:

$$\begin{aligned} T(c_1\mathbf{e}_1 + c_2\mathbf{e}_2) &= c_1T(\mathbf{e}_1) + c_2T(\mathbf{e}_2) \\ &= c_1(3\mathbf{e}_1) + c_2(1\mathbf{e}_2). \end{aligned}$$

In other words,  $T$  stretches by a factor of 3 in the  $\mathbf{e}_1$  direction, and by a factor of 1 in the  $\mathbf{e}_2$  direction, transforming a square grid in the domain into a parallel rectangular grid in the image:



Exercise 1) Do a similar geometric analysis and sketch for the transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Exercise 2) And for the transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Definition: If  $A_{n \times n}$  and if  $A \underline{v} = \lambda \underline{v}$  for a scalar  $\lambda$  and a vector  $\underline{v} \neq \underline{0}$  then  $\underline{v}$  is called an eigenvector of  $A$ , and  $\lambda$  is called the eigenvalue of  $\underline{v}$ . (In some texts the words characteristic vector and characteristic value are used as synonyms for these words.)

- In the three examples above, the standard basis vectors (or multiples of them) were eigenvectors, and the corresponding eigenvalues were the diagonal matrix entries. A non-diagonal matrix may still have eigenvectors and eigenvalues, and this geometric information can still be important to find. But how do you find eigenvectors and eigenvalues for non-diagonal matrices? ...

## How to find eigenvalues and eigenvectors (including eigenspaces):

If

$$\begin{aligned} A \underline{v} &= \lambda \underline{v} \\ \Leftrightarrow A \underline{v} - \lambda \underline{v} &= \underline{0} \\ \Leftrightarrow A \underline{v} - \lambda I \underline{v} &= \underline{0} \end{aligned}$$

where  $I$  is the identity matrix.

$$\Leftrightarrow (A - \lambda I) \underline{v} = \underline{0}.$$

As we know, this last equation can have non-zero solutions  $\underline{v}$  if and only if the matrix  $(A - \lambda I)$  is not invertible, i.e.

$$\Leftrightarrow \det(A - \lambda I) = 0.$$

So, to find the eigenvalues and eigenvectors of matrix you can proceed as follows:

- Compute the polynomial in  $\lambda$

$$p(\lambda) = \det(A - \lambda I).$$

If  $A_{n \times n}$  then  $p(\lambda)$  will be degree  $n$ . This polynomial is called the characteristic polynomial of the matrix  $A$ .

- $\lambda_j$  can be an eigenvalue for some non-zero eigenvector  $\underline{v}$  if and only if it's a root of the characteristic polynomial, i.e.  $p(\lambda_j) = 0$ . For each such root, the homogeneous solution space of vectors  $\underline{v}$  solving

$$(A - \lambda_j I) \underline{v} = \underline{0}$$

will be eigenvectors with eigenvalue  $\lambda_j$ . This subspace of eigenvectors will be at least one dimensional, since  $(A - \lambda_j I)$  does not reduce to the identity and so the explicit homogeneous solutions will have free parameters. Find a basis of eigenvectors for this subspace. Follow this procedure for each eigenvalue, i.e. for each root of the characteristic polynomial.

Notation: The subspace of eigenvectors for eigenvalue  $\lambda_j$  is called the  $\lambda_j$  eigenspace, and denoted by  $E_{\lambda_j}$ .

(We include the zero vector in  $E_{\lambda_j}$ .) The basis of eigenvectors is called an eigenbasis for  $E_{\lambda_j}$ .

Exercise 3) Verify that if we follow this procedure for the diagonal matrix in Exercise 1 then the eigenvalues are just the diagonal entries and the eigenspaces are the spans of the standard basis vectors. Notice that this would happen for any diagonal matrix.

Exercise 4) a) Find the eigenvalues and eigenbases for the non-diagonal matrix

$$A = \begin{bmatrix} 7 & 4 \\ -2 & 1 \end{bmatrix}.$$

b) Use your work to describe the geometry of the linear transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 7 & 4 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Exercise 5) Find the eigenvalues and eigenspace bases for

$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}.$$

- (i) Find the characteristic polynomial and factor it to find the eigenvalues.
- (ii) for each eigenvalue, find bases for the corresponding eigenspaces.

Your solution will be related to the output below:

```
> with(LinearAlgebra) :  
  B := Matrix(3, 3, [4, -2, 1, 2, 0, 1, 2, -2, 3]);  
  Eigenvectors(B);
```

$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

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