Math 2250-4
Fri Apr 20
9.2-9.3 Nonlinear systems of first order differential equations, with an emphasis on autonomous systems of two first order DEs. Linearization at equilibrium points and stability.

We're discussing autonomous systems of two first order differential equations for $x(t), y(t)$:

$$x' = F(x, y)$$
$$y' = G(x, y)$$

We're interested in equilibrium solutions, stability, linearization, and the connections to phase portraits.

• On Wednesday we linearized a competition model from ecology in order to understand solutions near an interesting equilibrium point. We did the linearization "the long way", and were about to discuss the shortcut that uses Calculus. Let's finish that discussion and the further exercises in Wednesday's notes.

• As we work through examples, we will be understanding that there is a fairly short list of possibilities for how the linearized solutions behave. Except in borderline cases (see Theorem on following pages) the stability properties and geometric pictures for the linearized DE's govern what happens in the non-linear problem as well.

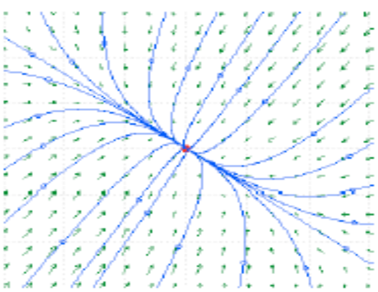
• In case the matrix A for the linearized system is <u>diagonalizable with real number eigenvalues</u>, the linearized system for $\underline{z}(t) = [u(t), v(t)]^T$

$$\underline{z}'(\underline{t}) = A\,\underline{z}$$

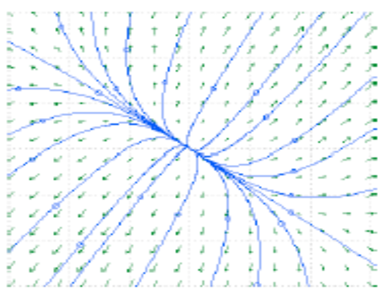
has general solution

$$\underline{z}(t) = c_1 e^{\lambda_1 t}\underline{v}_1 + c_2 e^{\lambda_2 t}\underline{v}_2 .$$

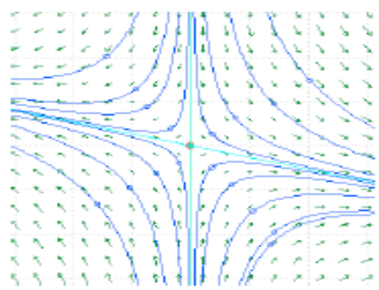
If each eigenvalue is non-zero, the three possibilities are:



nodal sink
$\lambda_1, \lambda_2 < 0$



nodal source
$\lambda_1, \lambda_2 > 0$



saddle point
$\lambda_1 < 0 < \lambda_2$

<u>Exercise 1)</u> Review how the two eigenvectors of the matrix A in the linear system show up in the three phase portraits above. (By the way, in the nodal cases, if the eigenvalues are equal then all solution trajectories follow rays from the origin, and the nodes are called "proper" nodes, as opposed to the case with unequal eigenvalues, in which case they're called "improper" nodes.

Remark: In case $A_{2 \times 2}$ is not diagonalizable, then the single eigenvalue λ_1 must be real, and its eigenspace must only be one-dimensional. In this case one can check that the general solution to the linearized system is

$$\underline{\boldsymbol{z}}(t) = c_1 e^{\lambda_1 t}\, \underline{\boldsymbol{v}} + c_2 e^{\lambda_1 t} \left(\underline{\boldsymbol{u}} + t\, \underline{\boldsymbol{v}}\right)$$

where $\underline{\boldsymbol{v}}$ is the eigenvector for λ_1 and $\left(A - \lambda_1 I\right)\underline{\boldsymbol{u}} = \underline{\boldsymbol{v}}$. Qualitatively the phase portraits look like those of improper stable or unstable nodes, depending on whether $\lambda_1 < 0$ or $\lambda_1 > 0$.

Complex roots!

Exercise 2) Consider this predator-prey model from section 9.3, for interacting populations $x(t), y(t)$. The population $x(t)$ is the prey (e.g. rabbits), and the population $y(t)$ is the predator (e.g. foxes).

$$x'(t) = 10\,x - x^2 - 5\,x\,y$$
$$y'(t) = -5\,y + x\,y.$$

2a) Discuss the various terms in this autonomous system of DEs. In this case there is good justification to use terms proportional to $x\,y$ to model how these populations interact. Explain.

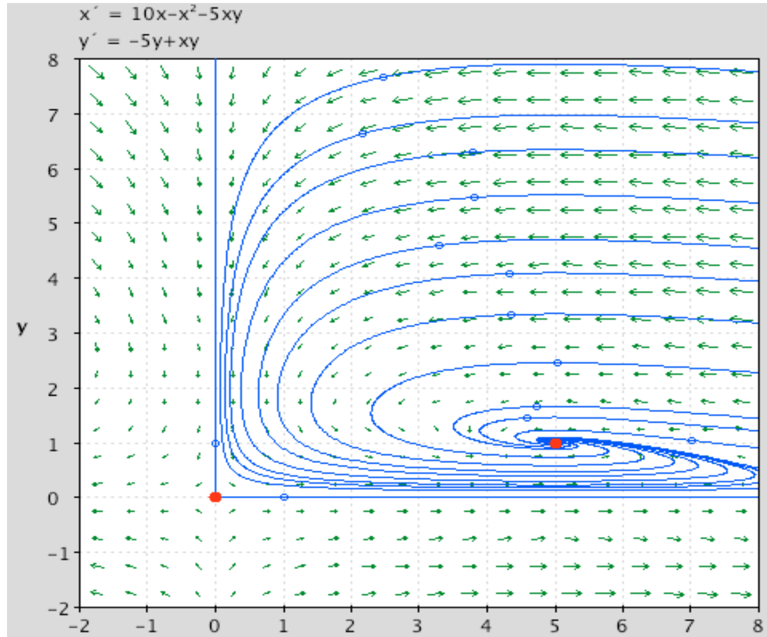
2b) Find the equilibrium solutions, classify the two with real eigenvalues, and sketch the phase portrait for the linearized problem.

2c) The equilbrium solution in the first quadrant is especially interesting. Linearize the system at this point and find the eigenvalues. They will be complex....and we know this leads to spiral phase diagrams (or elliptical if the eigenvalues are purely imaginary.) Rather than doing the painful computation of the general solution to the linearized system

$$\underline{\boldsymbol{z}}'(t) = A\,\underline{\boldsymbol{z}}$$

plot several values of the tangent vector field along the u, v axes to deduce whether the spirals are clockwise or counterclockwise, and their rough shape.

2d) Compare your work to the pplane phase portrait on the next page, and interpret your work in terms of the predator-prey model.



<u>General discussion of complex eigenvalues</u> for linear homogenous systems of two first order DE's: Let $A_{2 \times 2}$ have complex eigenvalues $\lambda = p \pm q\,i$. For $\lambda = p + q\,i$ let the eigenvector be $\underline{\boldsymbol{v}} = \underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\,i$. Then we know that we can use the complex solution $e^{\lambda t}\underline{\boldsymbol{v}}$ to extract two real vector-valued solutions, by taking the real and imaginary parts:

$$\begin{aligned} e^{\lambda t}\underline{\boldsymbol{v}} &= e^{(p+q\,i)t}(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\,i) \\ &= e^{p\,t}(\cos(q\,t) + i\sin(q\,t))(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\,i) \\ &= \left[e^{p\,t}\cos(q\,t)\underline{\boldsymbol{a}} - e^{p\,t}\sin(q\,t)\underline{\boldsymbol{b}}\right] \\ &\quad + i\left[e^{p\,t}\sin(q\,t)\underline{\boldsymbol{a}} + e^{p\,t}\cos(q\,t)\underline{\boldsymbol{b}}\right]. \end{aligned}$$

Thus, the general real solution is

$$\begin{aligned} \underline{\boldsymbol{z}}(t) &= c_1 e^{p\,t}[\cos(q\,t)\underline{\boldsymbol{a}} - \sin(q\,t)\underline{\boldsymbol{b}}] \\ &\quad + c_2 e^{p\,t}[\sin(q\,t)\underline{\boldsymbol{a}} + \cos(q\,t)\underline{\boldsymbol{b}}]. \end{aligned}$$

We can cleverly write this as a product of matrices times a vector as follows, and by looking carefully at this expression we can explain why we get spiral solutions when $\Re(\lambda) = p \neq 0$, and elliptical solutions when $p = 0$.

$$\begin{aligned} \underline{\boldsymbol{z}}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} &= e^{p\,t}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} c_1\cos(q\,t) \\ -c_1\sin(q\,t) \end{bmatrix} \\ &\quad + e^{p\,t}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} c_2\sin(q\,t) \\ c_2\cos(q\,t) \end{bmatrix} \\ &= e^{p\,t}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\left(\begin{bmatrix} c_1\cos(q\,t) \\ -c_1\sin(q\,t) \end{bmatrix} + \begin{bmatrix} c_2\sin(q\,t) \\ c_2\cos(q\,t) \end{bmatrix}\right) \end{aligned}$$

$$\underline{\boldsymbol{z}}(t) = e^{p\,t}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} \cos(q\,t) & \sin(q\,t) \\ -\sin(q\,t) & \cos(q\,t) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Exercise 3a) Check that the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

rotates vectors in the plane by an angle θ. The easiest way to verify this is to show that the standard basis vectors $e_1 = [1, 0]^T$ and $e_2 = [0, 1]^T$ are transformed correctly.

3b) Therefore

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \cos(q\,t) & \sin(q\,t) \\ -\sin(q\,t) & \cos(q\,t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

rotates $[c_1, c_2]^T$ by an angle $\theta = -q\,t$ and so as t varies, this part of the solution formula traces out a parametric circle of radius $\sqrt{c_1^2 + c_2^2}$.

3c) Therefore

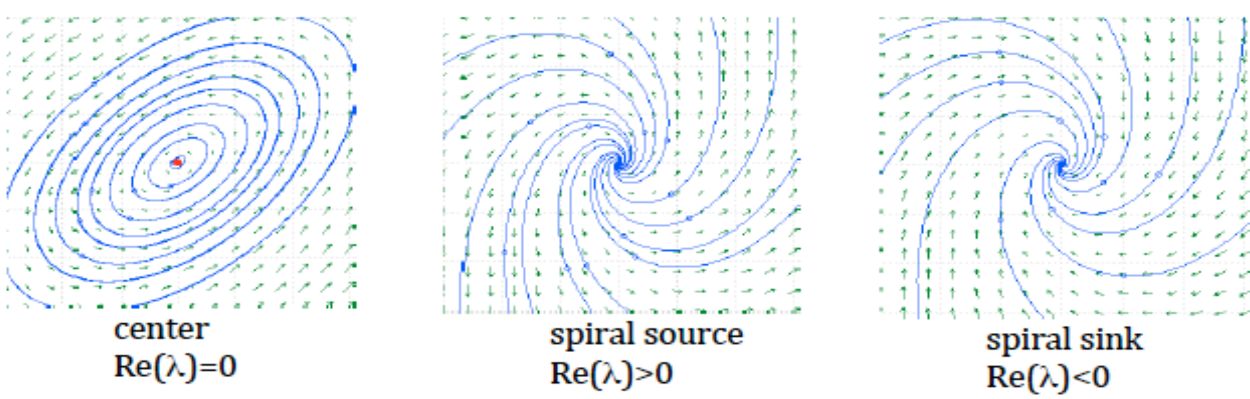
$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \cos(q\,t) & \sin(q\,t) \\ -\sin(q\,t) & \cos(q\,t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = y_1(t) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + y_2(t) \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

traces out a curve that looks like an ellipse (and is in fact an ellipse).

3d) Thus

$$\underline{z}(t) = e^{p\,t} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \cos(q\,t) & \sin(q\,t) \\ -\sin(q\,t) & \cos(q\,t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

traces out a stable spiral ("spiral sink") if $p < 0$, and unstable spiral ("spiral source") if $p > 0$, and an ellipse ("stable center") if $p = 0$.



Theorem: Let $[x_*, y_*]$ be an equilibrium point for a first order autonomous system of differential equations.

(i) If the linerized system of differential equations at $[x_*, y_*]$ has real eigendata, and either of an (asymptotically stable) nodal sink, an (unstable) nodal source, or an (unstable) saddle point, then the equilibrium solution for the non-linear system inherits the same stability and geometric properties as the linearized solution.

(ii) If the linearized system has complex eigendata, and if $\Re(\lambda) \neq 0$, then the equilibrium solution for the non-linear system is also either an (unstable) spiral source or a (stable) spiral sink. If the linearization yields a (stable) center, then further work is needed to deduce stability properties for the nonlinear system.

Alternate explanation of spiral behavior (which lets us review phase-amplitude form):
We have the general solutions

$$\underline{z}(t) = c_1 e^{pt}[\cos(qt)\underline{\boldsymbol{a}} - \sin(qt)\underline{\boldsymbol{b}}] + c_2 e^{pt}[\sin(qt)\underline{\boldsymbol{a}} + \cos(qt)\underline{\boldsymbol{b}}].$$

Rewrite as

$$\underline{z}(t) = e^{pt}\{[c_1\cos(qt) + c_2\sin(qt)]\underline{\boldsymbol{a}} + [-c_1\sin(qt) + c_2\cos(qt)]\underline{\boldsymbol{b}}\}$$

Write the bracketed coefficient of $\underline{\boldsymbol{a}}$ in amplitude-phase form:

$$c_1\cos(qt) + c_2\sin(qt) = C\cos(qt - \alpha)$$

with, as we originally discussed for amplitude-phase form:

$$c_1 = C\cos(\alpha)$$
$$c_2 = C\sin(\alpha).$$

Thus, the bracketed coefficient of $\underline{\boldsymbol{b}}$ is

$$-c_1\sin(qt) + c_2\cos(qt) = -C\cos(\alpha)\sin(qt) + C\sin(\alpha)\cos(qt)$$
$$= -C\sin(qt - \alpha).$$

Thus

$$\underline{z}(t) = e^{pt}\{C\cos(qt - \alpha)\underline{\boldsymbol{a}} + C\sin(qt - \alpha)(-\underline{\boldsymbol{b}})\}$$

which explains the elliptical behavior when $p = 0$, and the spiraling behavior when $p \neq 0$.