Math 2250-4
Mon Apr 2
10.5, EP7.6

Today we focus on the unit impulse ("delta function") Laplace transform entry, and the periodic function table entry, from those shown below. On Friday we discussed using the unit step function to turn functions on and off. We also discussed convolution integrals, to find inverse Laplace transforms of products $F(s)G(s)$. And, at the end of today's notes we'll see (EP7.6) why that convolution formula is so important in applications.

$f(t)$ with $\lvert f(t)\rvert \le C e^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$	comments
$u(t-a)$ unit step function	$\frac{e^{-as}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\,u(t-a)$	$e^{-as}F(s)$	more complicated on/off
$\delta(t-a)$	e^{-as}	unit impulse/delta "function"
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	convolution integrals to invert Laplace transform products
$f(t)$ with period p	$\frac{1}{1-e^{-ps}}\int_0^p f(t)e^{-st}\,dt$	for periodic forcing which is not sinusoidal.

<u>Periodic functions:</u> Let $f(t)$ be periodic with period p, i.e. $f(t+p) = f(t)\ \forall\ t$.

Notice that we can decompose f into the sum of a function which equals f on the interval $[0, p)$ and is zero on $[p, \infty)$, with a function that is zero on $[0, p)$ and equals f on $[p, \infty)$:

$$\begin{aligned} f(t) &= f(t)(1 - u(t-p)) + u(t-p)f(t) \\ &= f(t)(1 - u(t-p)) + u(t-p)f(t-p) \end{aligned}$$

(since f is p-periodic).

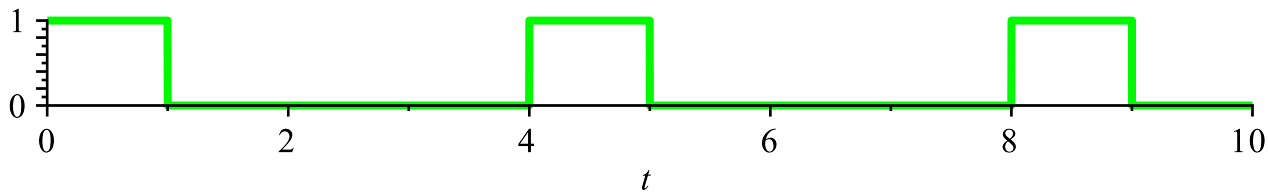
If we take the Laplace transform of this identity use the definition of Laplace transform for the first term on the right and the translation theorem for the second term, we get

$$F(s) = \int_0^p f(t)e^{-st}\,dt + e^{-ps}F(s)$$

$$+ \Rightarrow F(s)\left(1 - e^{-ps}\right) = \int_0^p f(t)e^{-st}\,dt$$

$$F(s) = \frac{1}{1-e^{-ps}}\int_0^p f(t)e^{-st}\,dt\ .$$

Exercise 1) Find the Laplace transform of the function whose graph is shown below. The function has period 4 :



(solution: $F(s) = \dfrac{1 - e^{-s}}{s\left(1 - e^{-4\,s}\right)}$)

EP 7.6 impulse functions and the δ operator.

Consider a force $f(t)$ acting on an object for only on a very short time interval $a \le t \le a+\varepsilon$, for example as when a bat hits a ball. This impulse p of the force is defined to be the integral

$$p := \int_a^{a+\varepsilon} f(t)\, dt$$

and it measures the net change in momentum of the object since by Newton's second law

$$m\, v'(t) = f(t)$$

$$\Rightarrow \int_a^{a+\varepsilon} m\, v'(t)\, dt = \int_a^{a+\varepsilon} f(t)\, dt = p$$

$$\Rightarrow m\, v(t)\Big]_{t=a}^{a+\varepsilon} = p\,.$$

Since the impulse p only depends on the integral of $f(t)$, and since the exact form of f is unlikely to be known in any case, the easiest model is to replace f with a constant force having the same total impulse, i.e. to set

$$f = p\, d_{a,\varepsilon}(t)$$

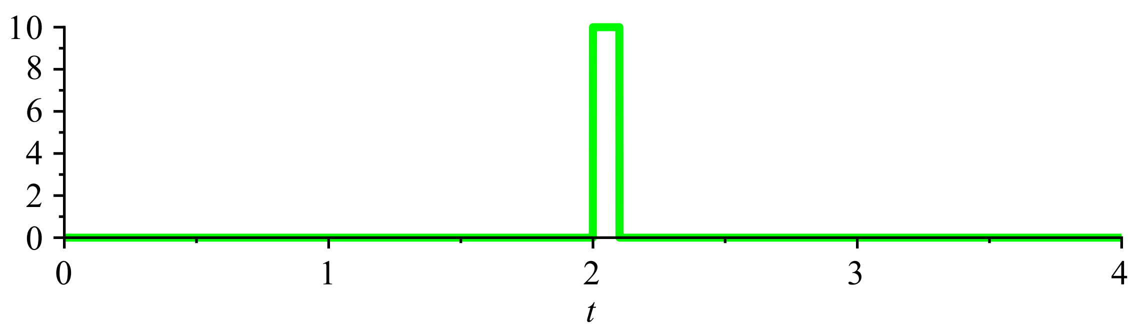
where $d_{a,\varepsilon}(t)$ is the unit impulse function given by

$$d_{a,\varepsilon}(t) = \begin{cases} 0, & t < a \\ \frac{1}{\varepsilon}, & a \le t < a+\varepsilon \\ 0, & t \ge a+\varepsilon \end{cases}.$$

Notice that

$$\int_a^{a+\varepsilon} d_{a,\varepsilon}(t)\, dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\, dt = 1\,.$$

Here's a graph of $d_{2,.1}(t)$, for example:



Since the unit impulse function is a linear combination of unit step functions, we could solve differential equations with impulse functions so-constructed. As far as Laplace transform goes, it's even easier to take the limit as $\varepsilon \to 0$ for the Laplace transforms $\mathcal{L}\{d_{a,\varepsilon}(t)\}(s)$, and this effectively models impulses on very short time scales.

$$d_{a,\varepsilon}(t) = \frac{1}{\varepsilon}\left[u(t-a) - u(t-(a+\varepsilon))\right]$$

$$\Rightarrow \mathcal{L}\{d_{a,\varepsilon}(t)\}(s) = \frac{1}{\varepsilon}\left(\frac{e^{-a s}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right)$$
$$= e^{-a s}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right).$$

In Laplace land we can use L'Hopital's rule (in the variable ε) to take the limit as $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} e^{-a s}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right) = e^{-a s}\lim_{\varepsilon \to 0}\left(\frac{s\, e^{-\varepsilon s}}{s}\right) = e^{-a s}.$$

The result in time t space is not really a function but we call it the "delta function" $\delta(t-a)$ anyways, and visualize it as a function that is zero everywhere except at $t = a$, and that it is infinite at $t = a$ in such a way that its integral over any open interval containing a equals one. As explained in EP7.6, the delta "function" can be thought of in a rigorous way as an operator, not as a function. It can also be thought of as the derivative of the unit step function $u(t-a)$, and this is consistent with the Laplace table entries for derivatives of functions. In any case, this leads to the very useful Laplace transform table entry

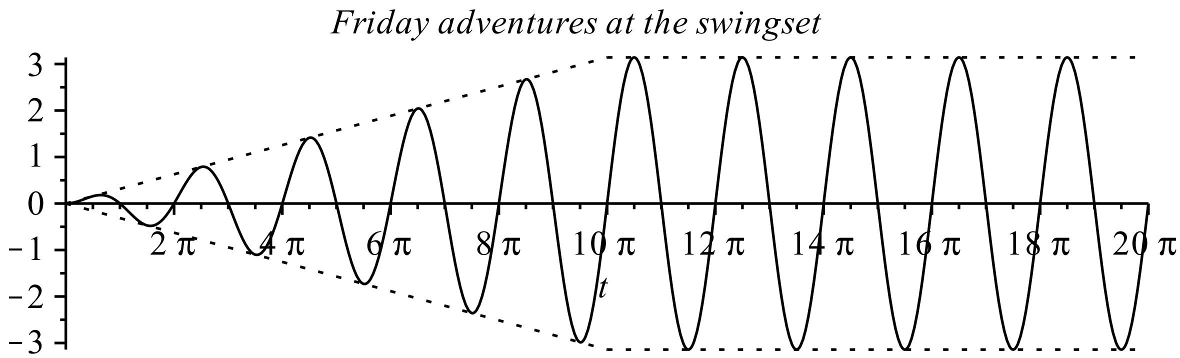
$\delta(t-a)$ unit impulse function	$e^{-a s}$	for impulse forcing

Exercise 2) Revisit the swing from Friday and solve the IVP below for $x(t)$. In this case the parent is providing an impulse each time the child passes through equilibrium position after completing a cycle.

$$x''(t) + x(t) = .2\,\pi[\delta(t) + \delta(t-2\pi) + \delta(t-4\pi) + \delta(t-6\pi) + \delta(t-8\pi)]$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

Impulse forcing of mechanical systems
EP 7.6

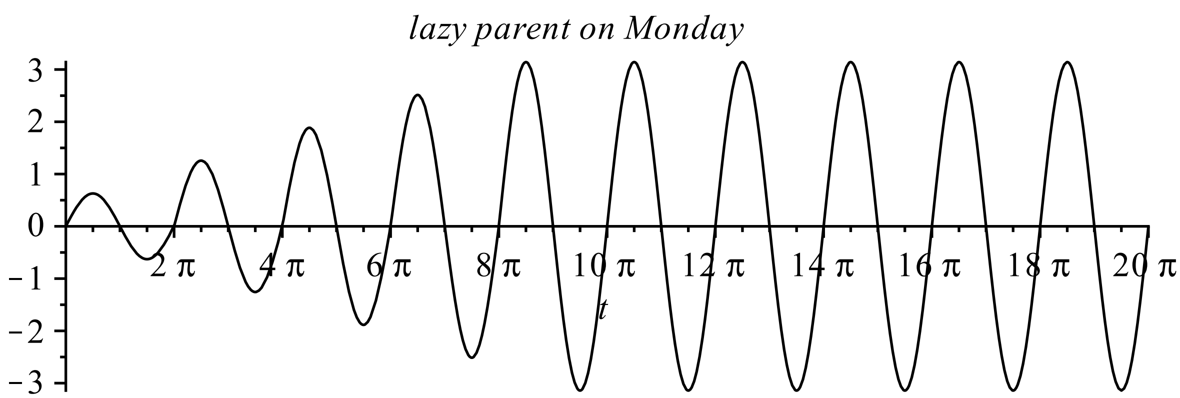
```
> with(plots) :
> plot1 := plot(.1·t·sin(t), t = 0 ..10·Pi, color = black) :
  plot2 := plot(Pi·sin(t), t = 10·Pi ..20·Pi, color = black) :
  plot3 := plot(Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot4 := plot(−Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot5 := plot(.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  plot6 := plot(−.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title = `Friday adventures at the swingset`);
```




```
>
```

impulse solution: five equal impulses to get same final amplitude of π meters - Exercise 2:

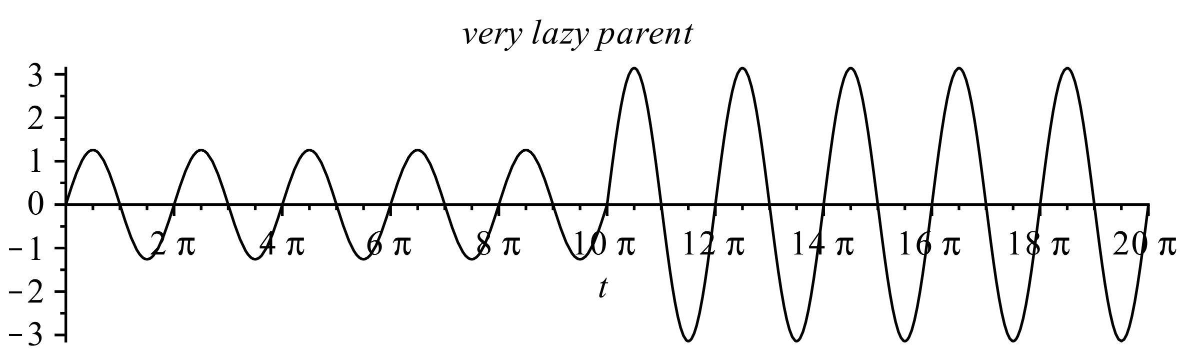
```
> f := t→.2·Pi·sum(Heaviside(t − k·2·Pi)·sin(t − k·2·Pi), k = 0 ..4) :
> plot(f(t), t = 0 ..20·Pi, color = black, title = `lazy parent on Monday`);
```




```
>
```

Or, an impulse at $t = 0$ and another one at $t = 10\,\pi$.

```
> g := t→.2·Pi·(2·sin(t) + 3·Heaviside(t − 10·Pi)·sin(t − 10·Pi)) :
> plot(g(t), t = 0 ..20·Pi, color = black, title = `very lazy parent`);
```




```
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```

Convolutions and solutions to non-homogeneous physical oscillation problems (EP7.6 p. 499-501)

Consider a mechanical or electrical forced oscillation problem for $x(t)$, and the particular solution that begins at rest:

$$a\,x'' + b\,x' + c\,x = f(t)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

Then in Laplace land, this equation is equivalent to

$$a\,s^2 X(s) + b\,s\,X(s) + c\,X(s) = F(s)$$
$$\Rightarrow X(s)\left(a\,s^2 + b\,s + c\right) = F(s)$$
$$\Rightarrow X(s) = F(s)\cdot\frac{1}{a\,s^2 + b\,s + c} := F(s)W(s)\,.$$

$\mathcal{L}^{-1}\{W(s)\}(t) = w(t)$ is called the weight function for the physical system, and because of the convolution table entry

$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	convolution integrals to invert Laplace transform products

the solution is given by

$$x(t) = f*w(t) = w*f(t) = \int_0^t w(\tau)f(t-\tau)\,d\tau\,.$$

This idea generalizes to much more complicated mechanical and circuit systems, and is how engineers experiment mathematically with how proposed configurations will respond to various input forcing functions, once they figure out the weight function for their system.

Exercise 3. Let's play the resonance game, with non-sinusoidal forcing functions. We'll stick with our earlier swing, but consider various forcing periodic functions $f(t)$ that we haven't thought about before.

$$x''(t) + x(t) = f(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

a) Find the weight function $w(t)$.
b) Write down the solution formula for $x(t)$ as a convolution integral.

c) Play the resonance game on the following pages ...

We worked out that the solution to our DE IVP will be

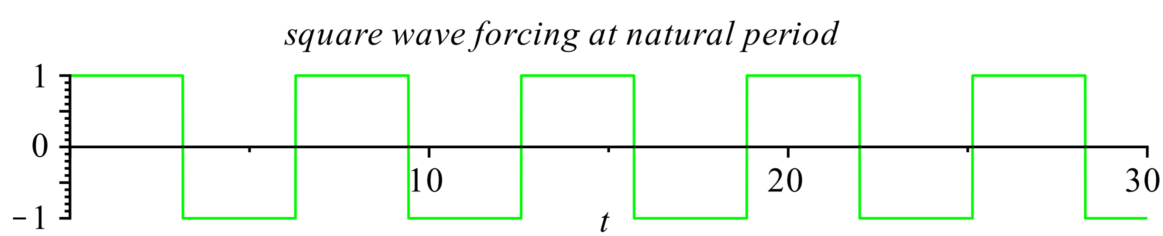
$$x(t) = \int_0^t \sin(\tau) f(t-\tau)\, d\tau$$

Since the unforced system has a natural angular frequency $\omega_0 = 1$, we expect resonance when the forcing function has the corresponding period of $T_0 = \frac{2\pi}{w_0} = 2\pi$. We will discover that there is the possibility for resonance if the period of f is a ***multiple*** of T_0. (Also, forcing at the natural period doesn't guarantee resonance...it depends what function you force with.)

<u>Example 1)</u> A square wave forcing function with amplitude 1 and period 2π. Let's talk about how we came up with the formula (which works until $t = 11\pi$) .

> $with(plots):$

> $f1 := t \to -1 + 2 \cdot \left(\sum_{n=0}^{10} (-1)^n \cdot \text{Heaviside}(t - n \cdot \text{Pi}) \right):$
$plot1a := plot(f1(t), t = 0..30, color = green):$
$display(plot1a, title = \text{`square wave forcing at natural period`});$



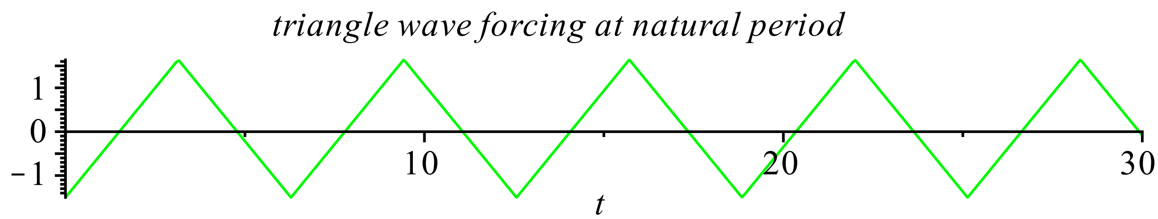
<u>1)</u> What's your vote? Is this square wave going to induce resonance, i.e. a response with linearly growing amplitude?

> $x1 := t \to \int_0^t \sin(\tau) \cdot f1(t-\tau)\, d\tau:$
$plot1b := plot(x1(t), t = 0..30, color = black):$
$display(\{plot1a, plot1b\}, title = \text{`resonance response ?`});$

Example 2) A triangle wave forcing function, same period

> $f2 := t \to \int_0^t f1(s)\,\mathrm{d}s - 1.5$:   # *this antiderivative of square wave should be triangle wave*

$plot2a := plot(f2(t), t = 0..30, color = green)$:
$display(plot2a, title = `triangle\ wave\ forcing\ at\ natural\ period`)$;



2) Resonance?

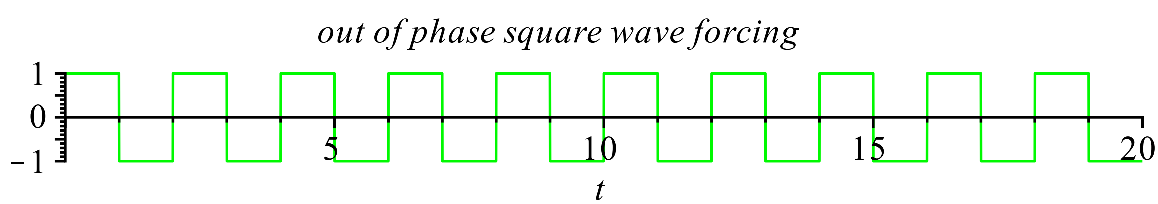
> $x2 := t \to \int_0^t \sin(\tau)\cdot f2(t-\tau)\,\mathrm{d}\tau$:

$plot2b := plot(x2(t), t = 0..30, color = black)$:
$display(\{plot2a, plot2b\}, title = `resonance\ response\ ?`)$;

Example 3) Forcing not at the natural period, e.g. with a square wave having period $T = 2$.

> $f3 := t \to -1 + 2\cdot \sum_{n=0}^{20} (-1)^n \cdot \mathrm{Heaviside}(t-n)$:

$plot3a := plot(f3(t), t = 0..20, color = green)$:
$display(plot3a, title = `out\ of\ phase\ square\ wave\ forcing`)$;



3) Resonance?

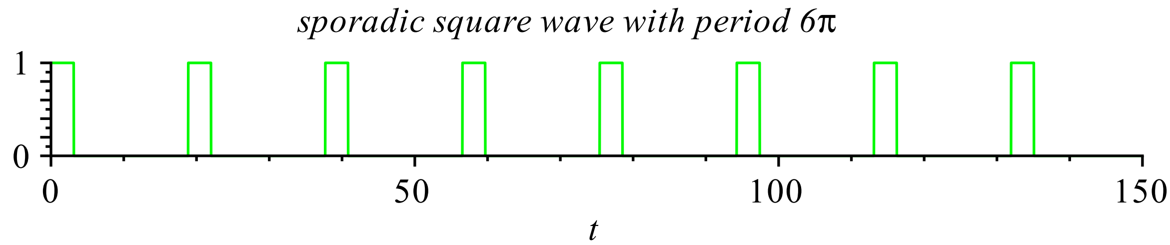
> $x3 := t \to \int_0^t \sin(\tau)\cdot f3(t-\tau)\,\mathrm{d}\tau$:

$plot3b := plot(x3(t), t = 0..20, color = black)$:
$display(\{plot3a, plot3b\}, title = `resonance\ response\ ?`)$;

Example 4) Forcing not at the natural period, e.g. with a particular wave having period $T = 6\pi$.

$$> f4 := t \rightarrow \sum_{n=0}^{10} \left(\text{Heaviside}(t - 6 \cdot n \cdot \pi) - \text{Heaviside}(t - (6 \cdot n + 1) \cdot \pi)\right) :$$

plot4a := *plot*(*f4*(*t*), *t* = 0 ..150, *color* = *green*) :
display(*plot4a*, *title* = `*sporadic square wave with period* 6π`);



4) Resonance?

$$> x4 := t \rightarrow \int_0^t \sin(\tau) \cdot f4(t - \tau) \, d\tau :$$

plot4b := *plot*(*x4*(*t*), *t* = 0 ..150, *color* = *black*) :
display({*plot4a*, *plot4b*}, *title* = `*resonance response ?*`);

>

Hey, what happened???? How do we need to modify our thinking if we force a system with something which is not sinusoidal, in terms of worrying about resonance? In the case that this was modeling a swing (pendulum), how is it getting pushed?

__Precise Answer:__ It turns out that any periodic function with period P is a (possibly infinite) superposition of a constant function with *cosine* and *sine* functions of periods $\left\{P, \frac{P}{2}, \frac{P}{3}, \frac{P}{4}, ...\right\}$. Equivalently, these functions in the superposition are

$\left\{1, \cos(\omega t), \sin(\omega t), \cos(2\,\omega t), \sin(2\,\omega t), \cos(3\,\omega t), \sin(3\,\omega t), ...\right\}$ with $\omega = \frac{2\cdot\pi}{P}$. This is the theory of Fourier series, which you will study in other courses, e.g. Math 3150, Partial Differential Equations. If the given periodic forcing function $f(t)$ has non-zero terms in this superposition for which $n\cdot\omega = \omega_0$ (the natural angular frequency) (equivalently $\frac{P}{n} = \frac{2\cdot\pi}{\omega_0}$), there will be resonance; otherwise, no resonance. We could already have understood some of this in Chapter 5, for example

__Exercise 4)__ The natural period of the following DE is (still) $T_0 = 2\,\pi$. Which of the two DE's below will have solutions that exhibit resonance? Use Chapter 5 superposition ideas. Compare your answer with how the period of the forcing function compares to the natural period of the system.

__a)__

$$x''(t) + x(t) = \cos(t) + \sin\left(\frac{t}{3}\right).$$

__b)__

$$x''(t) + x(t) = \cos(2\,t) - 3\,\sin(3\,t).$$