Math 2250-4
Wed Apr 18
9.1-9.2 Nonlinear systems of first order differential equations, with an emphasis on autonomous systems of two first order DEs.
Our first two pages today are the last two pages of yesterday's notes, in case you want to minimize printing.

The general system of two first order differential equations for $x(t)$, $y(t)$ can be written as
$$x'(t) = F(x(t), y(t), t)$$
$$y'(t) = G(x(t), y(t), t)$$
which we abbreviate as usual, by writing
$$x' = F(x, y, t)$$
$$y' = G(x, y, t) \ .$$
If the rates of change F, G only depend on the values of $x(t)$, $y(t)$ but not on t, i.e.
$$x' = F(x, y)$$
$$y' = G(x, y)$$
then the system is called <u>autonomous.</u> Autonomous systems are the focus of Chapter 9.

Constant solutions to autonomous systems are called <u>equilibrium solutions.</u> Thus, equilibrium solutions $x(t) \equiv x_*, y(t) \equiv y_*$ will correspond to solutions $\begin{bmatrix} x_*, y_* \end{bmatrix}^T$ to the nonlinear algebraic system
$$F(x, y) = 0$$
$$G(x, y) = 0$$

<u>Exercise 1)</u> Consider the "competing species" model from 9.2, shown below. For example and in appropriate units, $x(t)$ might be a squirrel population and $y(t)$ might be a rabbit population, competing on the same island sanctuary.
$$x'(t) = 14 x - 2 x^2 - x y$$
$$y'(t) = 16 y - 2 y^2 - x y \ .$$
<u>1a)</u> Notice that if either population is missing, the other population satisfies a logistic DE. Discuss how the signs of third terms on the right sides of these DEs indicate that the populations are competing with each other (rather than, for example, acting in symbiosis, or so that one of them is a predator of the other).
<u>1b)</u> Find the four equilibrium solutions to this competition model, algebraically.

- Equilibrium solutions $[x_*, y_*]^T$ to first order autonomous systems
$$x' = F(x, y)$$
$$y' = G(x, y)$$
are called <u>stable</u> if solutions to IVPs starting close (enough) to $[x_*, y_*]^T$ stay as close as desired.
- Equilibrium solutions are <u>unstable</u> if they are not stable.
- Equilibirum solutions $[x_*, y_*]^T$ are called <u>asymptotically stable</u> if they are stable and furthermore, IVP solutions that start close enough to $[x_*, y_*]^T$ converge to $[x_*, y_*]^T$ as $t \to \infty$.

<u>1c)</u> Discuss the pplane phase portrait below for the competition model on the previous page. Discuss the sorts of stability/instability exhibited by the four equilibrium solutions.
<u>1d)</u> Where are the two phase diagrams from Chapter 2, for the logistic behavior when there is only one population, in the phase portrait below?
<u>1e)</u> If both initial populations are positive, what do you expect their limiting populations will be?

$x' = 14x - 2x^2 - xy$
$y' = 16y - 2y^2 - xy$

8.4741, 10.928
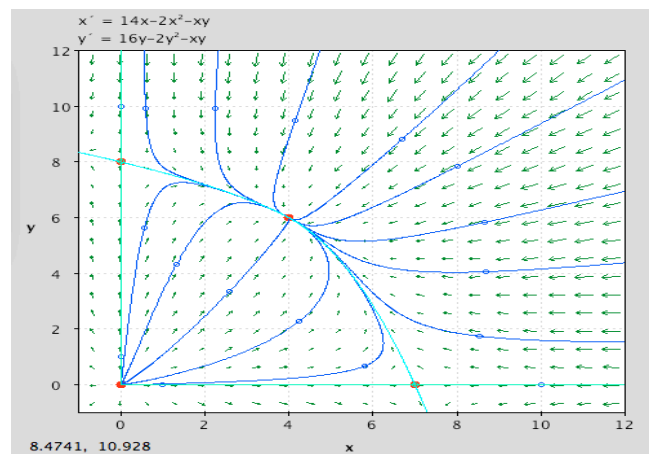
<u>Linearization</u> near equilibrium solutions is a recurring theme in differential equations, as we've already seen in this course. We can linearize autonomous systems of differential equations near equilibrium solutions. There is a fairly quick systematic way to do this, which we'll get to later, after we do a special case of linearization "the long way".

<u>Exercise 2)</u> We will linearize the rabbit-squirrel (competition) model of the previous example, near the equilibrium solution $[4, 6]^T$. Set

$$x(t) = 4 + u(t)$$
$$y(t) = 6 + v(t)$$

Thus, $[x, y]^T$ near $[4, 6]^T$ is equivalent to saying $[u, v]^T$ is near $[0, 0]$.

<u>2a)</u> Make the substitutions above into the original non-linear system

$$x'(t) = 14\,x - 2\,x^2 - x\,y$$
$$y'(t) = 16\,y - 2\,y^2 - x\,y$$

and use algebra to show that the deviations $u(t)$, $v(t)$ from 4, 6 respectively, satisfy

$$u'(t) = -8\,u - 4\,v - 2\,u^2 - uv$$
$$v'(t) = -6\,u - 12\,v - 2\,v^2 - uv.$$

When we "linearize" we assume $|u|$, $|v|$ are so small that the higher order terms are negligable. Thus the linearized system for the deviations $u(t), v(t)$ is

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ -6 & -12 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$$

And the solutions to this linear system of DE's should be very close to the solutions to the nonlinear system for the exact deviations from equilibrium, as long as $u(t), v(t)$ are each close enough to 0.

The matrix above has approximate eigendata:

$$\lambda_1 \approx -4.7, \quad \underline{v}_1 \approx [.79, -.64]^T$$
$$\lambda_2 \approx -15.3, \quad \underline{v}_2 \approx [.49, .89]^T$$

<u>2b)</u> Use the eigendata above to write down the general solution to the homogeneous (linearized) system. Use this information to make a rough sketch of the solution trajectories to the linearized problem near $[u, v]^T = [0, 0]^T$. Then compare your work to the pplane output on the next page. Notice how your phase portrait for the linearized problem near $[u, v]^T = [0, 0]^T$ looks very much like the phase portrait for $[x, y]^T$ near $[4, 6]^T$. This is sensible, since the correspondence is a translation of coordinate axes,

$$x = 4 + u$$
$$y = 6 + v$$

Linearization allows us to approximate and understand solutions to non-linear problems near equilbria:

The non-linear problem and representative solution curves:

$x' = 14x-2x^2-xy$
$y' = 16y-2y^2-xy$

9.9458, 5.0979
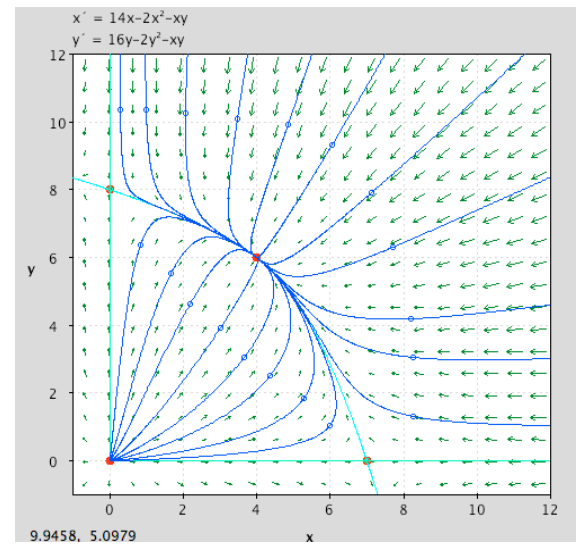
pplane will do the eigenvalue-eigenvector linearization computation for you - the matrix in the linear problem is called the Jacobian matrix:

Equilbrium Point:
   There is a nodal sink at (4, 6)
   Jacobian:
       -8   -4
       -6   -12
   The eigenvalues and eigenvectors are:
       -4.7085   (0.77218, -0.63541)
       -15.292   (0.48097, 0.87674)

The solutions to the linearized system near $[u, v]^T = [0, 0]^T$ are close to the exact solutions for non-linear deviations, so under the translation of coordinates $x = x_* + u$, $y = y_* + v$ the phase portrait for the linearized system looks like the phase portrait for the non-linear system.

$u' = -8u-4v$
$v' = -6u-12v$

0.82567, -0.41753
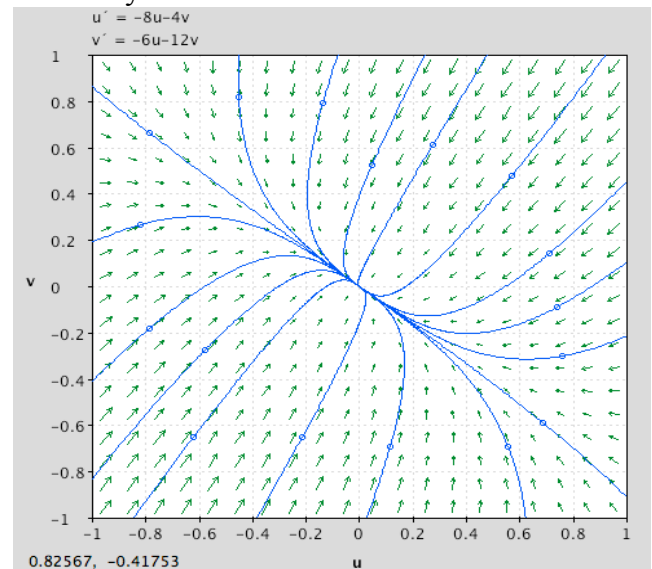
How to linearize with Calculus: (This would work for systems of n autonomous first order differential equations, but we focus on $n = 2$.)

$$x'(t) = F(x, y)$$
$$y'(t) = G(x, y)$$

Let $x(t) \equiv x_*, y(t) \equiv y_*$ be an equilibrium solution, i.e.

$$F(x_*, y_*) = 0$$
$$G(x_*, y_*) = 0 .$$

For solutions $[x(t), y(t)]^T$ to the original system, define the deviations from equilibrium $u(t), v(t)$ by

$$x(t) := x_* + u(t)$$
$$y(t) := y_* + v(t) .$$

Thus

$$u' = x' = F(x_* + u, y_* + v)$$
$$v' = y' = G(x_* + u, y_* + v) .$$

Using partial derivatives, which measure rates of change in the coordinate directions, we can approximate

$$u' = F(x_* + u, y_* + v) = F(x_*, y_*) + \frac{\partial F}{\partial x}(x_*, y_*) u + \frac{\partial F}{\partial y}(x_*, y_*) v + \varepsilon_1(u, v)$$

$$v' = G(x_* + u, y_* + v) = G(x_*, y_*) + \frac{\partial G}{\partial x}(x_*, y_*) u + \frac{\partial G}{\partial y}(x_*, y_*) v + \varepsilon_2(u, v)$$

For differentiable functions, the error terms $\varepsilon_1, \varepsilon_2$ shrink more quickly than the linear terms, as $u, v \to 0$. Also, note that $F(x_*, y_*) = G(x_*, y_*) = 0$. Thus the linearized system, written in vector-matrix form, is

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial x}(x_*, y_*) & \frac{\partial F}{\partial y}(x_*, y_*) \\ \frac{\partial G}{\partial x}(x_*, y_*) & \frac{\partial G}{\partial y}(x_*, y_*) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} .$$

The matrix of partial derivatives is called the Jacobian matrix. Notice that it is evaluated at the equilibrium point. We will often use the subscript notation for partial derivatives to save writing, e.g F_x for $\frac{\partial F}{\partial x}$ and F_y for $\frac{\partial F}{\partial y}$.

Exercise 3a) Redo the linearization computation for the competition model
$$x'(t) = 14 x - 2 x^2 - x y$$
$$y'(t) = 16 y - 2 y^2 - x y$$
at the equilibrium point $[4, 6]^T$, using the Jacobian matrix. It should be much quicker than the direct algebraic method.

3b) Then linearize at the equilibrium solution $[0, 8]^T$, find the general solution to the linearized system, and compare your work to the pplane picture for the original nonlinear system.