Math 2250-4
Tues Apr 17
7.4 Mass-spring systems.
Forced oscillations and inferring practical resonance in slightly damped problems. Application to models for earthquake-induced building vibrations.

If we have time I'll introduce Chapter 9 on autonomous nonlinear systems of differential equations. Sections 9.1-9.4 will be our final unit in this course, and will tie together many of the key ideas.

- As we voted yesterday, we will start class at the usual time of 10:45, even though the official end-time for the UU "shakeout" drill is 11:00.

- At the end of class yesterday we'd just finished finding a formula for a particular solution $\underline{\boldsymbol{x}}_P$ to the sinusoidally forced mechanical system equation

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}} + \cos(\omega\, t)\underline{\boldsymbol{F}}_0 .$$

It was

$$\underline{\boldsymbol{x}}_P(t) = \cos(\omega\, t)\underline{\boldsymbol{c}}$$

with

$$\underline{\boldsymbol{c}} = \underline{\boldsymbol{c}}(\omega) = -\left(A + \omega^2 I\right)^{-1}\underline{\boldsymbol{F}}_0 ,$$

as long as $\omega^2 \neq -\lambda_j$ for any of the eigenvalues of $A$. (In the cases $\omega = \sqrt{-\lambda_j}$ resonance would be likely, depending on whether or not the intermediate matrix system

$$\left(A + \omega^2 I\right)\underline{\boldsymbol{c}} = \underline{\boldsymbol{F}}_0$$

can be solved even though the inverse matrix doesn't exist.)

Use this result to complete Exercise 4 and the discussion of practical resonance that follows, in Tuesday's notes. Also discuss how these ideas apply to transverse oscillations as well as longitudinal ones, which is the final page in Tuesday's notes.

Then we can discuss the earthquake project (modified from the one in the text for section 7.4), which ties in to these ideas. The next two pages show the text version. Then I'll explain your (optional) project....it should be posted by some time tonight (finally).

## 7.4 Application Earthquake-Induced Vibrations of Multistory Buildings

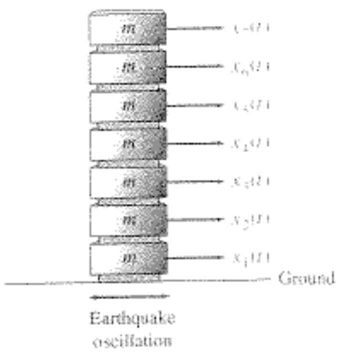


**FIGURE 7.4.15.** The seven-story building.

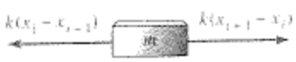


**FIGURE 7.4.16.** Forces on the $i$th floor.

In this application you are to investigate the response to transverse earthquake ground oscillations of the seven-story building illustrated in Fig. 7.4.15. Suppose that each of the seven (above-ground) floors weighs 16 tons, so the mass of each is $m = 1000$ (slugs). Also assume a horizontal restoring force of $k = 5$ (tons per foot) between adjacent floors. That is, the internal forces in response to horizontal displacements of the individual floors are those shown in Fig. 7.4.16. It follows that the free transverse oscillations indicated in Fig. 7.4.15 satisfy the equation $\mathbf{M}\mathbf{x}'' = \mathbf{K}\mathbf{x}$ with $n = 7$, $m_i = 1000$ (for each $i$), and $k_i = 10000$ (lb/ft) for $1 \leqq i \leqq 7$. The system then reduces to the form $\mathbf{x}'' = \mathbf{A}\mathbf{x}$ with

$$\mathbf{A} = \begin{bmatrix} -20 & 10 & 0 & 0 & 0 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & -20 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & -20 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & -20 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 & -20 & 10 \\ 0 & 0 & 0 & 0 & 0 & 10 & -10 \end{bmatrix}. \tag{1}$$

Once the matrix **A** has been entered, the TI-86 command `eigVl A` takes about 15 seconds to calculate the seven eigenvalues shown in the $\lambda$-column of the table in Fig. 7.4.17. Alternatively, you can use the *Maple* command `eigenvals(A)`, the MATLAB command `eig(A)`, or the *Mathematica* command `Eigenvalues[A]`.

step 1) understand the model above. (Note, the Maple commands are out of date.)
step 2) understand the table below:

| $i$ | Eigenvalue $\lambda$ | Frequency $\omega = \sqrt{-\lambda}$ | Period $P = \frac{2\pi}{\omega}$ (sec) |
|---|---|---|---|
| 1 | −38.2709 | 6.1863 | 1.0157 |
| 2 | −33.3826 | 5.7778 | 1.0875 |
| 3 | −26.1803 | 5.1167 | 1.2280 |
| 4 | −17.9094 | 4.2320 | 1.4847 |
| 5 | −10.0000 | 3.1623 | 1.9869 |
| 6 | −3.8197 | 1.9544 | 3.2149 |
| 7 | −0.4370 | 0.6611 | 9.5042 |

**FIGURE 7.4.17.** Frequencies and periods of natural oscillations of the seven-story building.

step 3) understand the inhomogeneous DE below in (2), and why the forcing term has the claimed form.
step 4) understand the practical resonance graph, figure 7.4.18, and how it relates to the natural periods in Figure 7.4.17.

Then calculate the entries in the remaining columns of the table showing the natural frequencies and periods of oscillation of the seven-story building. Note that a typical earthquake producing ground oscillations with a period of 2 seconds is uncomfortably close to the fifth natural frequency (with period 1.9869 seconds) of the building.

A horizontal earthquake oscillation $E\cos\omega t$ of the ground, with amplitude $E$ and acceleration $a = -E\omega^2\cos\omega t$, produces an opposite inertial force $F = ma = mE\omega^2\cos\omega t$ on each floor of the building. The resulting nonhomogeneous system is

try $x_p = \cos\omega t\ \vec{c}$ ← unknown.

$$\mathbf{x}'' = \mathbf{A}\mathbf{x} + (E\omega^2\cos\omega t)\mathbf{b}, \qquad (2)$$

where $\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$ and $\mathbf{A}$ is the matrix of Eq. (1). Figure 7.4.18 shows a plot of *maximal amplitude* (for the forced oscillations of any single floor) versus the *period* of the earthquake vibrations. The spikes correspond to the first six of the seven resonant frequencies. We see, for instance, that whereas an earthquake with period 2 (s) likely would produce destructive resonance vibrations in the building, it probably would be unharmed by an earthquake with period 2.5 (s). Different buildings have different natural frequencies of vibration, and so a given earthquake may demolish one building but leave untouched the one next door. This seeming anomaly was observed in Mexico City after the devastating earthquake of September 19, 1985.

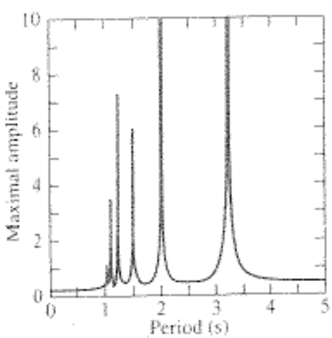


**FIGURE 7.4.18.** Resonance vibrations of a seven-story building—maximal amplitude as a function of period.

For your personal seven-story building to investigate, let the weight (in tons) of each story equal the largest digit of your student ID number and let $k$ (in tons/ft) equal the smallest nonzero digit. Produce numerical and graphical results like those illustrated in Figs. 7.4.17 and 7.4.18. Is your building susceptible to likely damage from an earthquake with period in the 2- to 3-second range?

You might like to begin by working manually the following warm-up problems.

1. Find the periods of the natural vibrations of a building with two above-ground floors, each weighing 16 tons and with each restoring force being $k = 5$ tons/ft.

In your Maple project you will be working with a 4-story building. In addition to finding the 4 natural frequencies and periods, and a practical resonance graph analogous to 7.4.18, you will describe the four fundamental modes of vibration. You may use Maple, Matlab, Mathematica, Wolfram alpha - whatever software you prefer. You will upload your results to CANVAS.

Here is a demonstration video of what happens for buildings with one, two, or three stories. The driving frequency is steadily increased, and various modes are excited (but not all of them).

http://www.youtube.com/watch?v=M_x2jOKAhZM modeling the 3 story building, as in the youtube video:

The ground is moving sinusoidally with amplitude $E$,

$$x_0(t) = E\cos(\omega t)$$

and acceleration

$$x_0''(t) = -E\,\omega^2\cos(\omega t)$$

the accelerations of the three massive floors (the top one is the roof) above ground, relative to the frame of the moving ground are therefore given by

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{bmatrix} = \frac{k}{m}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + E\omega^2\cos(\omega t)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Note the $-1$ value in the last diagonal entry. This is because $x_3(t)$ is measuring displacements for the top floor (roof), which has nothing above it.

Exercise i) do you understand how that inhomogeneous system of DE's arises?

Here is eigendata for the unscaled matrix $\left(\frac{k}{m} = 1\right)$. Maple does not realize the eigenvalues are real numbers, and so several extra commands are needed to extract clear information.

```
> with(LinearAlgebra) :
> A := Matrix(3, 3, [-2, 1.0, 0, 1, -2, 1, 0, 1, -1]);
      # I used one decimal value so Maple would evaluate in floating point
   Eigenvectors(A); # execute these commands to see the mess.
> Digits := 5 : # 5 digits should be fine, for one thing.
```

clean up the mess:

```
> eigendata := Eigenvectors(A) :
   eigendata[1] : #eigenvalues
   eigendata[2] : #corresponding eigenvectors
   lambdas := map(ℜ, eigendata[1]) : # get rid of the 0 i's by taking real part.
   omegas := map(sqrt, -lambdas); # natural angular frequencies
   f := x → 2·evalf(Pi)/x :
   periods := map(f, omegas); #natural periods
   vectors1 := map(ℜ, eigendata[2]) : # eigenvectors for fundamental modes
   vectors := map(evalf, vectors1); # get the digits down to 5
```

$$omegas := \begin{bmatrix} 1.8019 \\ 1.2470 \\ 0.44504 \end{bmatrix}$$

$$periods := \begin{bmatrix} 3.4870 \\ 5.0386 \\ 14.118 \end{bmatrix}$$

$$vectors := \begin{bmatrix} 0.59101 & 0.73698 & 0.32799 \\ -0.73698 & 0.32799 & 0.59101 \\ 0.32799 & -0.59101 & 0.73698 \end{bmatrix} \quad \textbf{(1)}$$

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Exercise ii) Interpret the data above, in terms of the natural modes for the shaking building (when $\frac{k}{m} = 1$). In the youtube video the first mode to appear is the slow and dangerous "sloshing mode", where all three floors oscillate in phase, with amplitude ratios 33 : 59 : 74 from the first to the third floor. What's the second mode that gets excited? The third mode?

Exercise iii) Could you modify the matrix algebra, and then the commands in yesterday's notes, to create practical resonance graphs as we did there?

Introduction to Chapter 9: Autonomous first order systems of differential equations

The general system of two first order differential equations for $x(t), y(t)$ can be written as

$$x'(t) = F(x(t), y(t), t)$$
$$y'(t) = G(x(t), y(t), t)$$

which we abbreviate as usual, by writing

$$x' = F(x, y, t)$$
$$y' = G(x, y, t) \ .$$

If the rates of change $F, G$ only depend on the values of $x(t), y(t)$ but not on $t$, i.e.

$$x' = F(x, y)$$
$$y' = G(x, y)$$

then the system is called autonomous. Autonomous systems are the focus of Chapter 9.

Constant solutions to autonomous systems are called equilibrium solutions. Thus, equilibrium solutions $x(t) \equiv x_*, y(t) \equiv y_*$ will correspond to solutions $[x_*, y_*]^T$ to the nonlinear algebraic system

$$F(x, y) = 0$$
$$G(x, y) = 0$$

Exercise 1) Consider the "competing species" model from 9.2, shown below. For example and in appropriate units, $x(t)$ might be a squirrel population and $y(t)$ might be a rabbit population, competing on the same island sanctuary.

$$x'(t) = 14\,x - 2\,x^2 - x\,y$$
$$y'(t) = 16\,y - 2\,y^2 - x\,y \ .$$

1a) Notice that if either population is missing, the other population satisfies a logistic DE. Discuss how the signs of third terms on the right sides of these DEs indicate that the populations are competing with each other (rather than, for example, acting in symbiosis, or so that one of them is a predator of the other).
1b) Find the four equilibrium solutions to this competition model, algebraically.

- Equilibrium solutions $[x_*, y_*]^T$ to first order autonomous systems

$$x' = F(x, y)$$
$$y' = G(x, y)$$

are called <u>stable</u> if solutions to IVPs starting close (enough) to $[x_*, y_*]^T$ stay as close as desired.

- Equilibrium solutions are <u>unstable</u> if they are not stable.
- Equilibirum solutions $[x_*, y_*]^T$ are called <u>asymptotically stable</u> if they are stable and furthermore, IVP solutions that start close enough to $[x_*, y_*]^T$ converge to $[x_*, y_*]^T$ as $t \to \infty$ .

<u>1c)</u> Discuss the pplane phase portrait below for the competition model on the previous page. Discuss the sorts of stability/instability exhibited by the four equilibrium solutions.
<u>1d)</u> Where are the two phase diagrams from Chapter 2, for the logistic behavior when there is only one population, in the phase portrait below?
<u>1e)</u> If both initial populations are positive, what do you expect their limiting populations will be?