Math 2250-4
Mon Apr 16
7.4 Mass-spring systems.
Finish Case 1: undamped and unforced oscillations.
Begin Case 2: undamped forced oscillations and infering practical resonance in slightly damped problems.

• Our first order of businesss is to decide how to deal with "shakeout day" tomorrow (Tuesday) morning, which begins at 10:15 a.m. and may impact our class because it is scheduled to last until 11:00 a. m.

• On Friday we considered conservative coupled mass spring systems with n masses, and discussed how to find bases for the 2n-dimensional solution spaces to the resulting system of n second order linear homogeneous differential equations
$$M \underline{x}''(t) = K \underline{x} \implies \underline{x}''(t) = A \underline{x}.$$
We focused on the case of 2 masses and three springs,
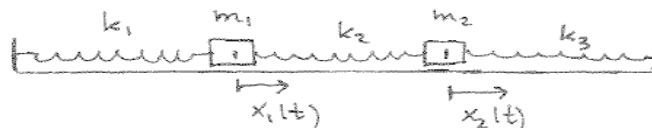
although in your homework you'll also consider configurations like these:
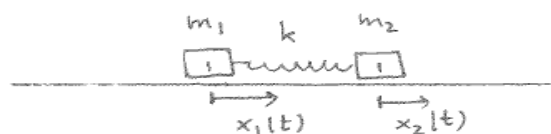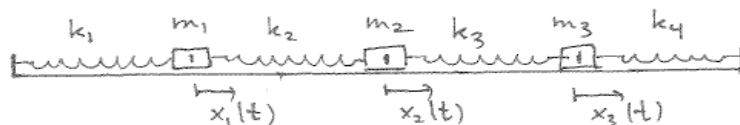
In the example case from Friday where all three Hooke's constants are the same, and the two mass values are the same, we found a basis for the 4-dimensional solution space, where each basis function was either $\cos\left(\omega_j t\right) \underline{v_j}$ or $\sin\left(\omega_j t\right) \underline{v_j}$, where the (two) $\underline{v_j}$'s were eigenvectors of A, with negative eigenvalues $\lambda_j = -\omega_j^2$ ($\omega_j = \sqrt{-\lambda}$).

Exercise 1) Return to Friday's notes, where we had just written down the solution at the top of page 4. Verify that you can solve each initial value problem for this second order system with a unique choice of the linear combination coefficients c_1, c_2, c_3, c_4. Then let's complete the experiment with the physics demo and see how accurate our modeling is!

<u>Exercise 2)</u> Consider a train with two cars connected by a spring:

m_1 k m_2

$x_1(t)$ $x_2(t)$
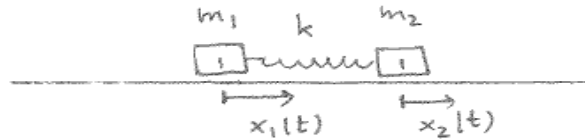
<u>2a)</u> Derive the linear system of DEs that governs the dynamics of this configuration (it's actually a special case of what we did before, with two of the spring constants equal to zero)

<u>2b).</u> Find the eigenvalues and eigenvectors. Then find the general solution. For $\lambda = 0$ and its corresponding eigenvector $\underline{v}$ verify that you get two solutions

$$\underline{x}(t) = \underline{v} \text{ and } \underline{x}(t) = t\,\underline{v},$$

rather than the expected $\cos(\omega t)\underline{v}$, $\sin(\omega t)\underline{v}$. Interpret these solutions in terms of train motions. You will use these ideas in some of your homework problems.

<u>Forced oscillations</u> (still undamped):
$$M \underline{x}''(t) = K \underline{x} + \underline{F}(t)$$
$$\Rightarrow \underline{x}''(t) = A \underline{x} + M^{-1} \underline{F}(t) .$$

If the forcing is sinusoidal,
$$M \underline{x}''(t) = K \underline{x} + \cos(\omega t) \underline{G}_0$$
$$\Rightarrow \underline{x}''(t) = A \underline{x} + \cos(\omega t) \underline{F}_0$$

with $\underline{F}_0 = M^{-1} \underline{G}_0$.

From the fundamental theorem for linear transformations we know that the general solution to this inhomogeneous linear problem is of the form
$$\underline{x}(t) = \underline{x}_P(t) + \underline{x}_H(t) ,$$
and we've just discussed how to find the homogeneous solutions $\underline{x}_H(t)$.

As long as the driving frequency ω is NOT one of the natural frequencies, we don't expect resonance; the method of undetermined coefficients predicts there should be a particular solution of the form
$$\underline{x}_P(t) = \cos(\omega t) \, \underline{c}$$
where the vector $\underline{c}$ is what we need to find.

<u>Exercise 3)</u> Substitute the guess $\underline{x}_P(t) = \cos(\omega t) \, \underline{c}$ into the DE system
$$\underline{x}''(t) = A \underline{x} + \cos(\omega t) \underline{F}_0$$
to find a matrix algebra formula for $\underline{c} = \underline{c}(\omega)$. Notice that this formula makes sense precisely when ω is NOT one of the natural frequencies of the system.

Solution:
$$\underline{c}(\omega) = -\left(A + \omega^2 I\right)^{-1} \underline{F}_0 .$$
Note, matrix inverse exists precisely if $-\omega^2$ is not an eigenvalue.

<u>Exercise 4)</u> Continuing with the example from Friday and today, and taking the case $k = m$, consider the special case of the previous page's discussion. Here we are forcing the second mass sinusoidally:

$$\begin{bmatrix} x_1{}''(t) \\ x_2{}''(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cos(\omega t) \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

We know from previous work that the natural frequencies are $\omega_1 = 1$, $\omega_2 = \sqrt{3}$ and that

$$\underline{x}_H(t) = C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\sqrt{3}\,t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find the formula for $\underline{x}_P(t)$, as on the preceding page. Notice that this steady periodic solution blows up as $\omega \to 1$ or $\omega \to \sqrt{3}$.

<u>Solution:</u> As long as $\omega \neq 1, \sqrt{3}$, the general solution $\underline{x} = \underline{x}_P + \underline{x}_H$ is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \cos(\omega t) \begin{bmatrix} \dfrac{3}{(\omega^2 - 1)(\omega^2 - 3)} \\ \dfrac{6 - 3\omega^2}{(\omega^2 - 1)(\omega^2 - 3)} \end{bmatrix}$$

$$+ \ C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\sqrt{3}\,t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

<u>Interpretation</u> as far as inferred practical resonance for slightly damped problems: If there was even a small amount of damping, the homogeneous solution would actually be transient (it would be exponentially decaying and oscillating - underdamped). There would still be a sinusoidal particular solution, which would have a formula close to our particular solution, the first term above, as long as $\omega \neq 1, \sqrt{3}$. (There would also be a relatively smaller $\sin(\omega t)\underline{d}$ term.) So we can infer the practical resonance behavior for different ω values with slight damping, by looking at the size of the $\underline{c}(\omega)$ term for the undamped problem....see next page for visualizations.

```
> restart :
> with(LinearAlgebra) :
> A := Matrix(2, 2, [−2, 1, 1, −2]) :
> F0 := Vector([0, 3]) :
> Iden := IdentityMatrix(2) :
> c := ω→(A + ω²·Iden)⁻¹.(−F0) :  # the formula we worked out by hand
> c(ω);
```
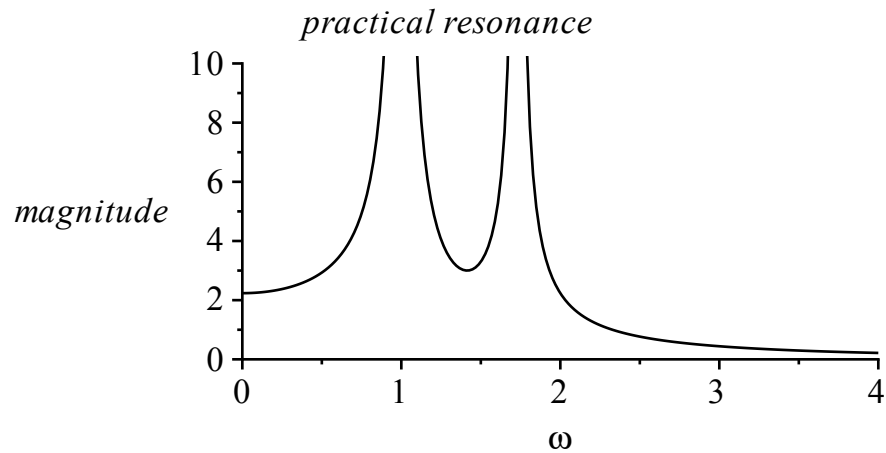
$$\begin{bmatrix} \dfrac{3}{3 - 4\,\omega^2 + \omega^4} \\[2ex] -\dfrac{3\left(-2 + \omega^2\right)}{3 - 4\,\omega^2 + \omega^4} \end{bmatrix}$$

(1)

```
> with(plots) :
  with(LinearAlgebra) :
> plot(Norm(c(ω), 2), ω = 0..4, magnitude = 0..10, color = black, title = `practical resonance`);
      # Norm(c(ω),2) is the magnitude of the c(ω) vector
```
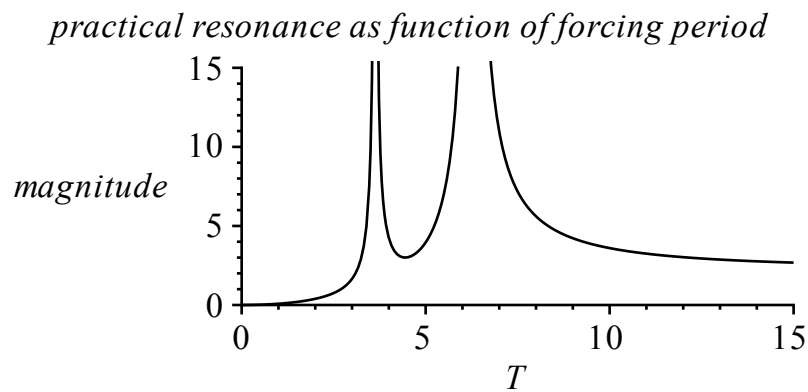
practical resonance
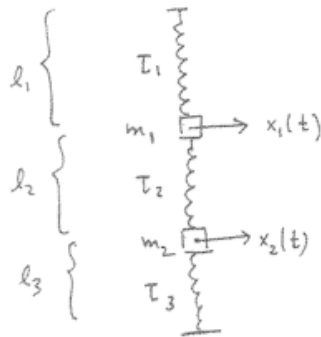


```
> plot( Norm( c( 2·Pi / T ), 2 ), T = 0..15, magnitude = 0..15, color = black, title
      = `practical resonance as function of forcing period` );
```

practical resonance as function of forcing period


```
>
```

There are strong connections between our discussion here and the modeling of how earthquakes can shake buildings:

- Transverse oscillations! (i.e. directions $\perp$ to the mass-spring configuration)

T_1, T_2, T_3 are the tensions (forces) of the stretched springs

By linearization, a good model would be

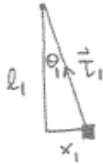$$m_1 x_1'' = -K_1 x_1 + K_2(x_2 - x_1) = -(K_1 + K_2)x_1 + K_2 x_2$$
$$m_2 x_2'' = K_2(x_1 - x_2) - K_3 x_2 = K_2 x_1 - (K_2 + K_3)x_2$$

where K_1, K_2, K_3 are positive constants as before

$\rightarrow$ but in general not the Hooke's constants, because to first order the springs are not being stretched beyond their equilibrium lengths in this model.

- upshot: transverse oscillations satisfy analogous systems of 2nd order linear DE's; forcing and resonance will also be analogous to longitudinal vibrations, but probably with different resonant frequencies & $\neq$ fundamental modes.

As it turns out, for our physics lab springs, the modes and frequencies are almost identical:

[force picture, e.g.

horiz force from top spring on mass 1
$$= -T_1 \sin\theta_1 = -T_1 \frac{x_1}{\sqrt{\ell_1^2 + x_1^2}} \approx -T_1 \frac{x_1}{\ell_1} = -\frac{T_1}{\ell_1} x_1$$

So $K_1 = \frac{T_1}{\ell_1}$

similarly, $K_2 = \frac{T_2}{\ell_2}$, $K_3 = \frac{T_3}{\ell_3}$

for our physics demo springs, equilibrium length ≈ 0, very Hookesian so $T \cong k\ell$; $\frac{T}{\ell} \approx k$, so actually almost recover same fundamental modes !!]

http://www.youtube.com/watch?v=M_x2jOKAhZM (an engineering class demo shake table)