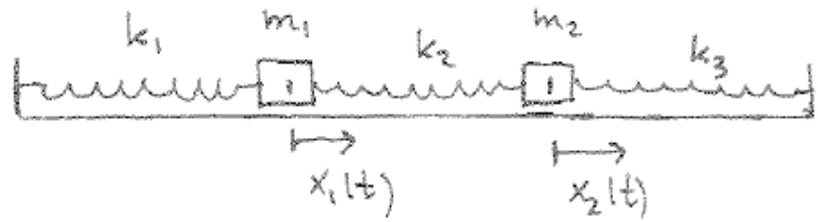
Math 2250-4
Fri Apr 13
7.4 Mass-spring systems. Case 1: undamped and unforced oscillations.

In your homework for today you modeled the spring system below, with no damping. Although we draw the picture horizontally, it would also hold in vertical configuration if we measure displacements from equilibrium in the underlying gravitational field. In fact, we'll do an experiment today with just such a vertical configuration.



Let's make sure we understand why the natural system of DEs and IVP for this system is

$$m_1 x_1''(t) = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2''(t) = -k_2(x_2 - x_1) - k_3 x_2$$
$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

Exercise 1a) What is the dimension of the solution space to this homogeneous linear system of differential equations? Why?

1b) What if one had a configuration of n masses in series, rather than just 2 masses? What would the dimension of the homogeneous solution space be in this case? Why?

We can write the system of DEs in matrix-vector form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We denote the diagonal matrix on the left as the "mass matrix" M, and the matrix on the right as the spring constant matrix K (although to be completely in sync with Chapter 5 it would be better to call it $-K$).

$$M\underline{\mathbf{x}}''(t) = K\underline{\mathbf{x}}.$$

If we divide each equation by the reciprocal of the corresponding mass, we can solve for the vector of accelerations:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2+k_3}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which we write as

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}.$$

(You can think of A as the "acceleration" matrix.)
Notice that the simplification above is mathematically identical to the algebraic operation of multiplying by preceding matrix equation by the (diagonal) inverse of the diagonal mass matrix M:

$$M\,\underline{\boldsymbol{x}}''(t) = K\,\underline{\boldsymbol{x}} \quad \Rightarrow \underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}\ \text{, with } A = M^{-1}K.$$

How to find a basis for the solution space to conserved-energy mass-spring systems of DEs

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}\ .$$

The simplest solutions would be of the form $f(t)\underline{\boldsymbol{v}}$, where $\underline{\boldsymbol{v}}$ is a constant vector. In the case of a single mass, we got simple harmonic motion spanned by sinusoidal functions $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$, where ω_0 depended on k, m. We first tried e^{rt} but Euler's formula led us to the trig functions. You can do a similar derivation here, starting with guesses of the form $e^{\mu t}\underline{\boldsymbol{v}}$, but let's just cut to the chase and try right away for solutions of the form

$$\cos(\omega t)\underline{\boldsymbol{v}} \quad \sin(\omega t)\underline{\boldsymbol{v}}\ .$$

If we substitute $\underline{\boldsymbol{x}}(t) = \cos(\omega t)\underline{\boldsymbol{v}}$ in the DE system

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}} \quad \Rightarrow -\omega^2\cos(\omega t)\underline{\boldsymbol{v}} = A(\cos(\omega t)\underline{\boldsymbol{v}}) = \cos(\omega t)A\,\underline{\boldsymbol{v}}\ .$$

This identity will hold $\forall\ t$ if and only if

$$A\,\underline{\boldsymbol{v}} = -\omega^2\underline{\boldsymbol{v}}.$$

So, $\underline{\boldsymbol{v}}$ must be an eigenvector of A, but its eigenvalue is $\lambda = -\omega^2$. If we used a trial solution $\underline{\boldsymbol{y}}(t) = \sin(\omega t)\underline{\boldsymbol{v}}$ we would arrive at the same eigenvector equation. This leads to the

Solution space algorithm: If $A_{n \times n}$ is a diagonalizable matrix and if all of its eigenvalues are negative, then for each eigenpair $(\lambda_j, \underline{\boldsymbol{v}}_j)$ there are two linearly independent solutions to $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$ given by

$$\underline{\boldsymbol{x}}_j(t) = \cos(\omega_j t)\underline{\boldsymbol{v}}_j \qquad \underline{\boldsymbol{y}}_j(t) = \sin(\omega_j t)\underline{\boldsymbol{v}}_j$$

with

$$\omega_j = \sqrt{-\lambda_j}\ .$$

This procedure constructs $2\,n$ independent solutions to the system $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$, i.e. a basis for the solution space.

Remark: What's amazing is that the fact that if the system is conservative the acceleration matrix will always be diagonalizable, and all of its eigenvalues will be non-positive. In fact, if the system is tethered to at least one wall (as in the diagram on page 1), all of the eigenvalues will be strictly negative, and the algorithm above will always yield a basis for the solution space. (If the system is not tethered and is free to move as a train, then $\lambda = 0$ will be one of the eigenvalues, and will yield the constant velocity contribution to the solution space, $(c_1 + c_2 t)\underline{\mathbf{v}}$, where $\underline{\mathbf{v}}$ is the corresponding eigenvector. Together with the solutions from strictly negative eigenvalues this will still lead to the general homogeneous solution.)

Exercise 2) Consider the special case of the configuration on page one for which $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. In this case, the equation for the vector of the two mass accelerations reduces to

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

a) Find the eigendata for the matrix

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

b) Deduce the eigendata for the acceleration matrix A which is $\frac{k}{m}$ times this matrix.

c) Find the $4-$ dimensional solution space to this two-mass, three-spring system.

<u>solution</u> The general solution is a superposition of two "fundamental modes". In the slower mode both masses oscillate "in phase", with equal amplitudes, and with angular frequency $\omega_1 = \sqrt{\frac{k}{m}}$. In the faster mode, both masses oscillate "out of phase" with equal amplitudes, and with angular frequency $\omega_2 = \sqrt{\frac{3\,k}{m}}$. The general solution can be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 \cos(\omega_1 t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\omega_2 t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_3 \cos(\omega_2 t) + c_4 \sin(\omega_2 t)\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The two mass, three spring system....Experiment!

Data: Each mass is 50 grams. Each spring mass is 10 grams. (Remember, and this is a defect, our model assumes massless springs.) The springs are "identical", and a mass of 50 grams stretches the spring 15.6 centimeters. (We should recheck this since it's last fall's data; we should also test the spring's "Hookesiness"). With the old numbers we get Hooke's constant

```
> Digits := 4:
> solve(k·.156 = .05·9.806, k)
```

$$3.143 \tag{1}$$

```
>
```

Here's Maple confirmation for some of our diligent work:

```
> with(LinearAlgebra):
  A := Matrix(2, 2, [-2·k/m, k/m, k/m, -2·k/m]);
  Eigenvectors(A);
```

$$A := \begin{bmatrix} -\frac{2\,k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2\,k}{m} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{3\,k}{m} \\ -\frac{k}{m} \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \tag{2}$$

Predict the two natural periods from the model and our experimental value of k, m. Then make the system vibrate in each mode individually and compare your prediction to the actual periods of these two fundamental modes.

ANSWER: If you do the model correctly and my office data is close to our class data, you will come up with theoretical natural periods of close to .46 and .79 seconds. I predict that the actual natural periods are a little longer, especially for the slow mode. (In my office experiment I got periods of 0.482 and 0.855 seconds.) What happened?

EXPLANATION: The springs actually have mass, equal to 10 grams each. This is almost on the same order of magnitude as the yellow masses, and causes the actual experiment to run more slowly than our model predicts. In order to be more accurate the total energy of our model must account for the kinetic energy of the springs. You actually have the tools to model this more-complicated situation, using the ideas of total energy discussed in section 5.6, and a "little" Calculus. You can carry out this analysis, like I sketched for the single mass, single spring oscillator (mar7.pdf), assuming that the spring velocity at a point on the spring linearly interpolates the velocity of the wall and mass (or mass and mass) which bounds it. It turns out that this gives an *A*-matrix the same eigenvectors, but different eigenvalues, namely

$$\lambda_1 = -\frac{6\,k}{6\,m + 5\,m_s}$$

$$\lambda_2 = -\frac{6\,k}{2\,m + m_s}.$$

(Hints: the "M" matrix is not diagonal but the "K" matrix is the same.)

If you use these values, then you get period predictions

```
> m := .05;
  ms := .010;
  k := 3.143;
  Omega1 := sqrt(6*k/(6*m + 5*ms));
  Omega2 := sqrt(6*k/(2*m + ms));
  T1 := evalf(2*Pi/Omega1);
  T2 := evalf(2*Pi/Omega2);
```

$$m := 0.05$$
$$ms := 0.010$$
$$k := 3.143$$
$$\Omega 1 := 7.340$$
$$\Omega 2 := 13.09$$
$$T1 := 0.8559$$
$$T2 := 0.4801 \qquad (3)$$

of .856 and .480 seconds per cycle. Is that closer?

Challenge: If you can construct (and explain to me in my office) a correct derivation of the eigenvalues /eigenvectors I claim above, by taking the spring masses into account, then you can replace your low homework score this term with "20".