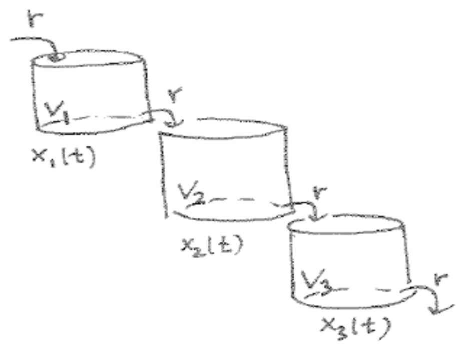
Math 2250-4
Wed Apr 11
7.3 continued....solutions to homogeneous and non-homogeneous linear systems of differential equations

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}} + \underline{\boldsymbol{f}}(t)$$

when the coefficient matrix A does not depend on time.

- Discuss the linearized glucose-insulin metabolism example in Tuesday's notes.
- Then consider the three component input-output model below:



Exercise 1a) Derive the first order system for the tank cascade above.

1b) In case the tank volumes (in gallons) are $V_1 = 20$, $V_2 = 40$, $V_3 = 50$, the flow rate $r = 10\ \frac{gal}{\min}$, and pure water (with no solute) is flowing into the first (top) tank, show that your system in (a) can be written as

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

(This system is actually worked out in the text, page 422-434...but we'll modify the IVP, and then consider a second case as well.)

1c) Here's the eigenvector data for the matrix in b. You may want to check or derive parts of it by hand, especially if you're still not expert at finding eigenvalues and eigenvectors. I entered the matrix entries as rational numbers rather than decimals, because otherwise Maple would have given (confusing) floating point answers. Use the eigendata to write down the general solution to the system in b.

```
> with(LinearAlgebra):
> A := Matrix(3, 3, [-1/2, 0, 0, 1/2, -1/4, 0, 0, 1/4, -1/5]);
  Eigenvectors(A);
```

$$A := \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{5} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{5} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & 0 & \frac{3}{5} \\ -\frac{1}{5} & 0 & -\frac{6}{5} \\ 1 & 1 & 1 \end{bmatrix} \quad (1)$$

```
>
```

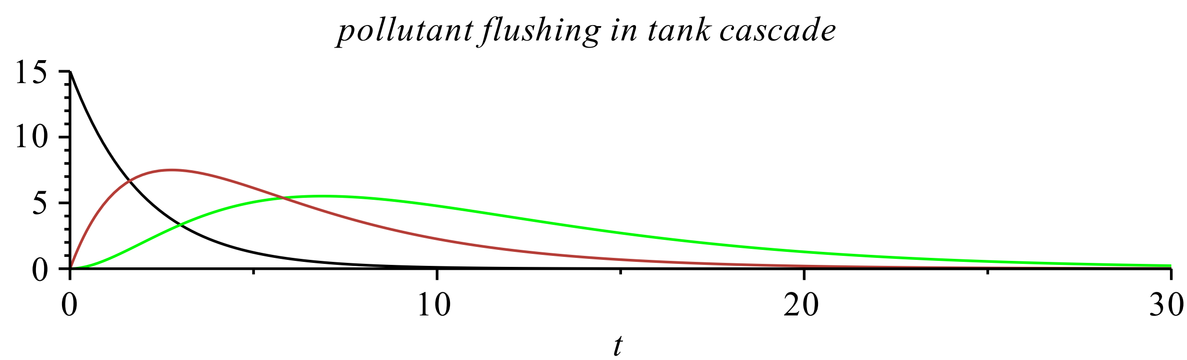
1d) Solve the IVP for this tank cascade, assuming that there are initially 15 *lb* of salt in the first tank, and no salt in the second and third tanks.

Your answer to d should be

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = 5\,\mathrm{e}^{-0.5\,t} \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} + 30\,\mathrm{e}^{-0.25\,t} \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} + 125\,\mathrm{e}^{-0.2\,t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

1e) We can plot the amounts of salt in each tank to figure out what's going on. Make sure you understand how the formulas below are related to the vector equation above, and interpret the graphical results.

```
> with(plots) :
> x1 := t→15·exp(−.5·t) :
  plot1 := plot(x1(t), t = 0 ..30, color = black) :
  x2 := t→ −30·exp(−.5·t) + 30·exp(−.25·t) :
  plot2 := plot(x2(t), t = 0 ..30, color = brown) :
  x3 := t→25·exp(−.5·t) − 150·exp(−.25·t) + 125·exp(−.2·t) :
  plot3 := plot(x3(t), t = 0 ..30, color = green) :
  display({plot1, plot2, plot3}, title = `pollutant flushing in tank cascade`);
```

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```

Exercise 2) Use the same tank cascade. Only now, assume that there is initially 13 *lb* salt in the first tank, none in the others, and that when the water starts flowing the input pipe contains salty water, with concentration $0.5\ \frac{lb}{gal}$.

2a) Explain why this yields an IVP for an inhomogeneous system of linear DE's, namely

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} - \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 13 \\ 0 \\ 0 \end{bmatrix}.$$

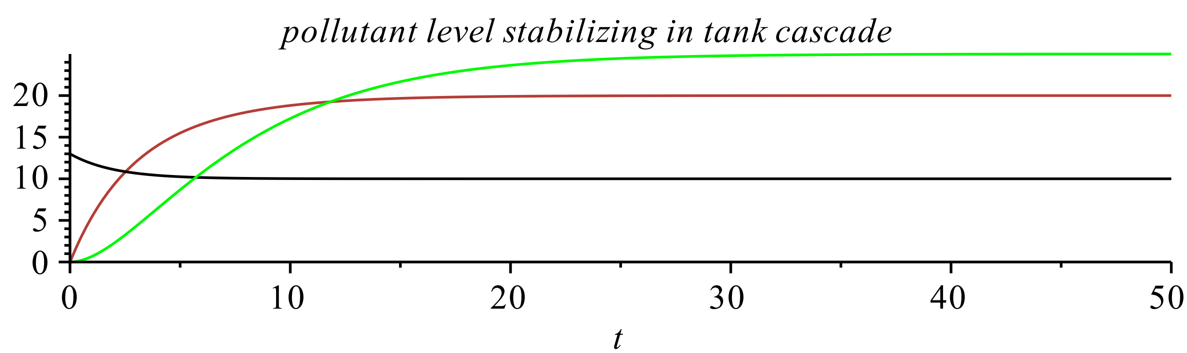
2b) Use a vector analog of "undetermined coefficients" to guess that there might be a particular solution that is a constant vector, i.e.

$$\underline{\boldsymbol{x}}_P(t) = \underline{\boldsymbol{c}}.$$

Plug this guess into the inhomgeneous system to deduce $\underline{\boldsymbol{c}}$. How could your irritating younger sibling have told you this particular solution, without knowing anything at all about linear algebra or differential equations?

2c) Use $\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}}_P(t) + \underline{\boldsymbol{x}}_H(t)$ to solve the IVP. Compare your solution to the plots below.

```
> x1 := t→10 + 3·exp(−.5·t) :
  plot1 := plot(x1(t), t=0 ..50, color=black) :
  x2 := t→20 − 6·exp(−.5·t)−14·exp(−.25·t) :
  plot2 := plot(x2(t), t=0 ..50, color=brown) :
  x3 := t→25 + 5·exp(−.5·t) + 70·exp(−.25·t) − 100·exp(−.2·t) :
  plot3 := plot(x3(t), t=0 ..50, color=green) :
  display({plot1, plot2, plot3}, title=`pollutant level stabilizing in tank cascade`);
```

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