

Math 2250-4

Tues Apr 10

7.3 Using eigenvectors to find the general solution to first order homogeneous systems of differential equations, in case the matrix A does not depend on time:

$$\underline{\mathbf{x}}'(t) = A \underline{\mathbf{x}}$$

• Recall that if $A_{n \times n}$ is diagonalizable, then we can find a basis of n solutions to the system above, of the form

$$\underline{\mathbf{x}}_j(t) = e^{\lambda_j t} \underline{\mathbf{v}}_j, \quad j = 1 \dots n$$

with $A \underline{\mathbf{v}}_j = \lambda_j \underline{\mathbf{v}}_j$.

The last example we worked on Monday was to use this method to find the general solution to

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We found eigenvalues and eigenvectors for the matrix as follows:

$$\lambda = -6, \quad \underline{\mathbf{v}} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$
$$\lambda = -1, \quad \underline{\mathbf{v}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

So the general solution to the system of DEs is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-6t} \begin{bmatrix} 1 \\ -6 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Exercise 1)

a) Solve the initial value problem for this system of DEs with

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

b) As it turns out, this first order system IVP is equivalent to a second order DE IVP for $x(t) := x_1(t)$. Derive this (overdamped) DE IVP. Use your work in a to read off the solution.

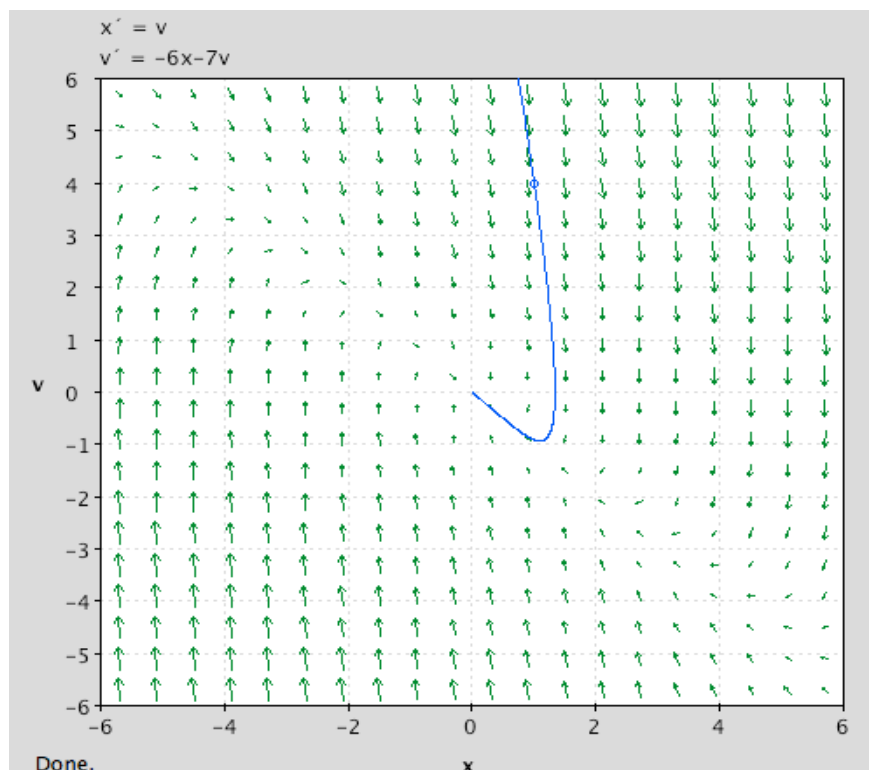
c) What do you notice about the Chapter 5 "characteristic polynomial" for $x(t)$ in b and the Chapter 6 "characteristic polynomial" for the matrix in the first order system?

d) What do you notice about the Chapter 5 "Wronskian matrix" for the second order DE, and the Chapter 7 "Wronskian matrix" for the solution to the first order system?

e) Since in the correspondence above, $x_2(t)$ equals the mass velocity $x'(t) = v(t)$, I've created the pplane phase portrait below using the lettering $[x(t), v(t)]^T$ rather than $[x_1(t), x_2(t)]^T$. Interpret the behavior of the overdamped mass-spring motion in terms of the pplane phase portrait.

f) How do the eigenvectors show up in the phase portrait, in terms of the direction the origin is approached from as $t \rightarrow \infty$, and the direction solutions came from (as $t \rightarrow -\infty$)?

<http://math.rice.edu/~dfield/dfpp.html>



So far we've not considered the possibility of complex eigenvalues and eigenvectors. Linear algebra theory works the same with complex number scalars and vectors - one can talk about complex vector spaces, linear combinations, span, linear independence, reduced row echelon form, determinant, dimension, basis, etc. Then the model space is \mathbb{C}^n rather than \mathbb{R}^n .

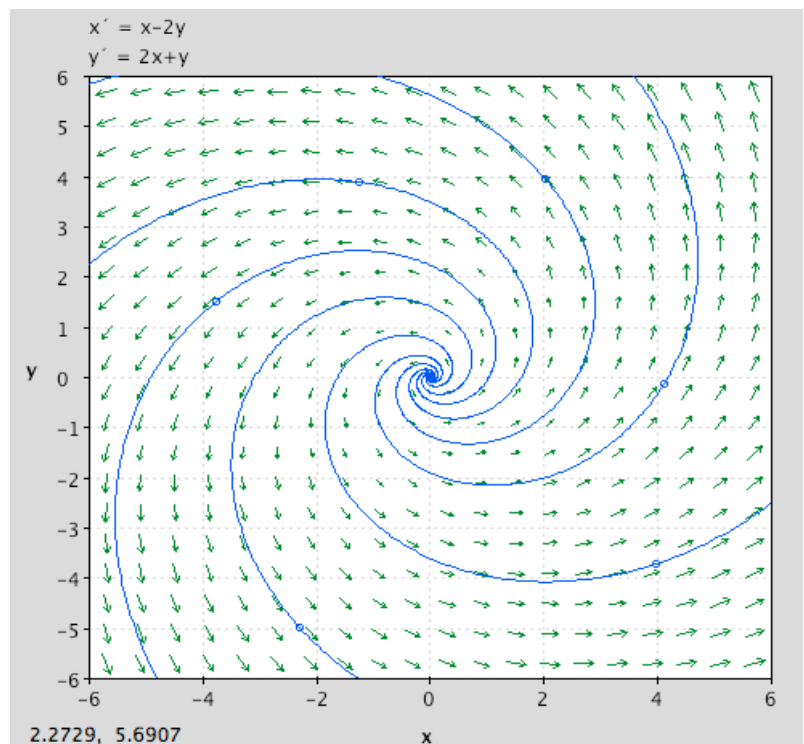
Definition: $\underline{v} \in \mathbb{C}^n$ ($\underline{v} \neq \underline{0}$) is a complex eigenvector of the matrix A , with eigenvalue $\lambda \in \mathbb{C}$ if $A \underline{v} = \lambda \underline{v}$.

Just as before, you find the possibly complex eigenvalues by finding the roots of the characteristic polynomial $|A - \lambda I|$. Then find the eigenspace bases by reducing the corresponding matrix (using complex scalars in the elementary row operations).

The best way to see how to proceed in the case of complex eigenvalues/eigenvectors is to work an example. We can also refer to the general discussion on the following page, at appropriate stages.

Exercise 2) Find the general solution to the following homogeneous system of DEs. Interpret your solution in terms of the pplane phase portrait shown below

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$



Solutions to homogeneous linear systems of DE's when matrix has complex eigenvalues:

$$\underline{\mathbf{x}}'(t) = A \underline{\mathbf{x}}$$

Let A be a real number matrix. Let

$$\lambda = a + b i \in \mathbb{C}$$

$$\underline{\mathbf{y}} = \underline{\alpha} + i \underline{\beta} \in \mathbb{C}^n$$

satisfy $A \underline{\mathbf{y}} = \lambda \underline{\mathbf{y}}$, with $a, b \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}^n$.

- Then $\underline{\mathbf{z}}(t) = e^{\lambda t} \underline{\mathbf{y}}$ is a complex solution to

$$\underline{\mathbf{x}}'(t) = A \underline{\mathbf{x}}$$

because $\underline{\mathbf{z}}'(t) = \lambda e^{\lambda t} \underline{\mathbf{y}}$ and this is equal to $A \underline{\mathbf{z}} = A e^{\lambda t} \underline{\mathbf{y}} = e^{\lambda t} A \underline{\mathbf{y}}$.

- But if we write $\underline{\mathbf{z}}(t)$ in terms of its real and imaginary parts,

$$\underline{\mathbf{z}}(t) = \underline{\mathbf{x}}(t) + i \underline{\mathbf{y}}(t)$$

then the equality

$$\begin{aligned} \underline{\mathbf{z}}'(t) &= A \underline{\mathbf{z}} \\ \Rightarrow \underline{\mathbf{x}}'(t) + i \underline{\mathbf{y}}'(t) &= A(\underline{\mathbf{x}}(t) + i \underline{\mathbf{y}}(t)) = A \underline{\mathbf{x}}(t) + i A \underline{\mathbf{y}}(t). \end{aligned}$$

Equating the real and imaginary parts on each side yields

$$\begin{aligned} \underline{\mathbf{x}}'(t) &= A \underline{\mathbf{x}}(t) \\ \underline{\mathbf{y}}'(t) &= A \underline{\mathbf{y}}(t) \end{aligned}$$

i.e. the real and imaginary parts of the complex solution are each real solutions.

- If $A(\underline{\alpha} + i \underline{\beta}) = (a + b i)(\underline{\alpha} + i \underline{\beta})$ then it is straightforward to check that $A(\underline{\alpha} - i \underline{\beta}) = (a - b i)(\underline{\alpha} - i \underline{\beta})$. Thus the complex conjugate eigenvalue yields the complex conjugate eigenvector. The corresponding complex solution to the system of DEs

$$e^{(a - i b)t}(\underline{\alpha} - i \underline{\beta}) = \underline{\mathbf{x}}(t) - i \underline{\mathbf{y}}(t)$$

so yields the same two real solutions (except with a sign change on the second one).

Here's a genuine first order system of DEs (that doesn't secretly come from a second order DE), and that shows the importance of being conversant with complex eigendata.

Glucose-insulin model (adapted from a discussion on page 339 of the text "Linear Algebra with Applications," by Otto Bretscher)

Let $G(t)$ be the excess glucose concentration (mg of G per 100 ml of blood, say) in someone's blood, at time t hours. Excess means we are keeping track of the difference between current and equilibrium ("fasting") concentrations. Similarly, Let $H(t)$ be the excess insulin concentration at time t hours. When blood levels of glucose rise, say as food is digested, the pancreas reacts by secreting insulin in order to utilize the glucose. Researchers have developed mathematical models for the glucose regulatory system. Here is a simplified (linearized) version of one such model, with particular representative matrix coefficients. It would be meant to apply between meals, when no additional glucose is being added to the system:

$$\begin{bmatrix} G'(t) \\ H'(t) \end{bmatrix} = \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

Exercise 3a) Understand why the signs of the matrix entries are reasonable.

Now let's solve the initial value problem, say right after a big meal, when

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

3b) The first step is to get the eigendata of the matrix. Do this, and compare with the Maple check on the next page.

```

[>
> with(LinearAlgebra) :
> A := Matrix(2, 2, [-1/10, -4/10, 1/10, -1/10]);
  Eigenvectors(A);

```

$$A := \begin{bmatrix} -\frac{1}{10} & -\frac{2}{5} \\ \frac{1}{10} & -\frac{1}{10} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{10} + \frac{1}{5} I \\ -\frac{1}{10} - \frac{1}{5} I \end{bmatrix}, \begin{bmatrix} 2 I & -2 I \\ 1 & 1 \end{bmatrix} \quad (1)$$

(2)

Notice that Maple writes a capital $I = \sqrt{-1}$.

3c) Extract a basis for the solution space to his homogeneous system of differential equations from the eigenvector information above:

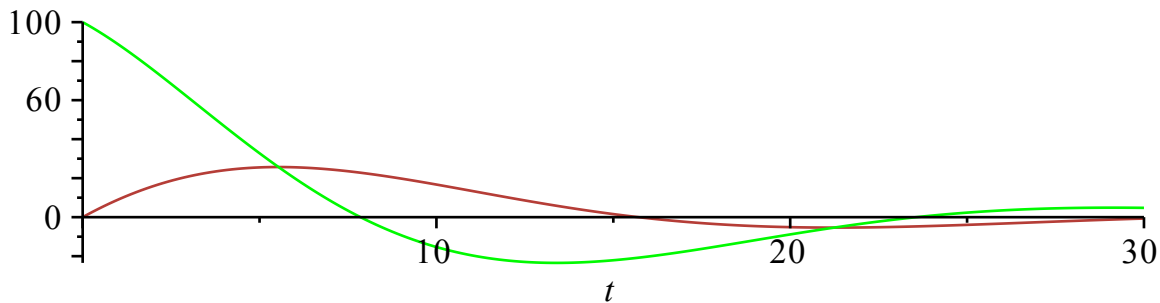
3d) Solve the initial value problem.

Here are some pictures to help understand what the model is predicting ... you could also construct these graphs using pplane, and take screen shots.

(1) Plots of glucose vs. insulin, at time t hours later:

```
> with(plots) :
> G := t → 100 · exp(−.1 · t) · cos(.2 · t) :
  H := t → 50 · exp(−.1 · t) · sin(.2 · t) :
  plot1 := plot(G(t), t = 0 .. 30, color = green) :
  plot2 := plot(H(t), t = 0 .. 30, color = brown) :
  display({plot1, plot2}, title = `underdamped glucose-insulin interactions`);
```

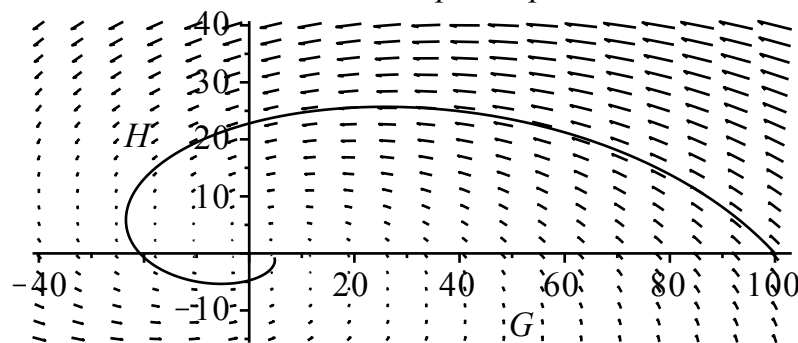
underdamped glucose-insulin interactions



2) A phase portrait of the glucose-insulin system:

```
> pict1 := fieldplot([−.1 · G − .4 · H, .1 · G − .1 · H], G = −40 .. 100, H = −15 .. 40) :
  soltn := plot([G(t), H(t), t = 0 .. 30], color = black) :
  display({pict1, soltn}, title = `Glucose vs Insulin phase portrait`);
```

Glucose vs Insulin phase portrait



- The example we just worked is linear, and is vastly simplified. But mathematicians, doctors, bioengineers, pharmacists, are very interested in (especially more realistic) problems like these. Prof. Fred Adler and graduate student Chris Remien in the Math Department, and collaborating with the University Hospital very recently modeled liver poisoning by acetomynaphen (brand name Tylenol), by studying a non-linear system of 8 first order differential equations. They came up with a state of the art and very useful diagnostic test:

http://unews.utah.edu/news_releases/math-can-save-tylenol-overdose-patients-2/

Here's a link to a preprint of their paper. For fun, I copied and pasted the system of differential equations from it, below:

<http://www.math.utah.edu/~korevaar/2250spring12/adler-remien-preprint.pdf>

APAP

$$\frac{dA}{dt} = -\frac{\alpha}{H_{max}} AH - \delta_a A$$

NAPQI

$$\frac{dN}{dt} = \frac{qp\alpha}{H_{max}} A - \gamma NG$$

GSH

$$\frac{dG}{dt} = \kappa - \gamma NG - \delta_g G$$

Functional Hepatocytes

$$\frac{dH}{dt} = rH \left(1 - \frac{H+Z}{H_{max}} \right) - \eta NH$$

Damaged Hepatocytes

$$\frac{dZ}{dt} = \eta NH - \delta_z Z$$

AST

$$\frac{dS}{dt} = \frac{d_z \beta_s}{\theta H_{max}} Z - \delta_s (S - S_{min})$$

ALT

$$\frac{dL}{dt} = \frac{d_z \beta_l}{\theta H_{max}} Z - \delta_l (L - L_{min})$$

Clotting Factor

$$\frac{dF}{dt} = \beta_f \left(\frac{H}{H_{max}} - F \right)$$