

Math 2250-4
Tuesday March 8
§5.3; begin §5.4

We're going through the algorithm to solve the constant coefficient linear homogeneous DE

$$\mathcal{L}(y) = 0$$

where $\mathcal{L}(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_2y'' + a_1y' + a_0y$. a_j 's constant.

In all cases, trying $y = e^{rx}$ leads to

$$\mathcal{L}(y) = (r^n + a_{n-1}r^{n-1} + \dots + a_1r + a_0)e^{rx}$$

so $\mathcal{L}(y) = 0$ exactly when r is a root of the charact poly $p(r)$.

We've understood

Case I: distinct real roots

Case II: repeated roots

and are discussing

Case III: complex roots.

Euler: $e^{-i\theta} = \cos\theta + i\sin\theta$

so, for $z = a + ib$, define

$$e^z := e^a e^{ib} = e^a (\cos b + i\sin b)$$

← we motivated this yesterday, via Taylor series

Def: if $f(x) + ig(x)$ is a complex-valued function of the real variable x ,

then $(f(x) + ig(x))' := f'(x) + ig'(x)$.

So if $\mathcal{L}$ is a linear differential operator with real coefficients, we can talk about complex solutions to

$$\mathcal{L}(y) = 0.$$

if $y = f + ig$, then $\mathcal{L}(y) = \mathcal{L}(f) + i\mathcal{L}(g)$, so $\mathcal{L}(y) = 0$ iff $\mathcal{L}(f) = 0$ and $\mathcal{L}(g) = 0$.

Theorem: If r is a complex root of the characteristic polynomial, then

$y = e^{rx}$ is a complex sol'n of $\mathcal{L}(y) = 0$.

why All we used to connect roots to solutions was

$$\frac{d}{dx} e^{rx} = r e^{rx}$$

since then

$$\mathcal{L}(e^{rx}) = p(r)e^{rx}.$$

Check box on next page!

Let $r = a + bi$

then $e^{rx} = e^{(a+bi)x} = e^{ax} (\cos bx + i \sin bx)$

$$\begin{aligned} \text{so } \frac{d}{dx}(e^{rx}) &= \frac{d}{dx}(e^{ax}(\cos bx + i \sin bx)) = e^{ax} [a(\cos bx + i \sin bx) + -b \sin bx + i b \cos bx] \\ &= e^{ax} [a \cos bx - b \sin bx + i(a \sin bx + b \cos bx)] \\ &\stackrel{?}{=} (a+bi) e^{ax} (\cos bx + i \sin bx) \quad \checkmark \\ &= r e^{rx} \end{aligned}$$

Thus, if r is a root of the characteristic poly, then

$$\mathcal{L}(e^{rx}) = p(r) e^{rx} = 0 = 0 + 0i$$

$$\mathcal{L}(e^{ax} \cos bx + i e^{ax} \sin bx)$$

$$\mathcal{L}(y_1(x)) + i \mathcal{L}(y_2(x)) \quad \text{by def of complex deriv.}$$

Equating real & imag. parts, deduce

$$\mathcal{L}(y_1) = 0, \quad \mathcal{L}(y_2) = 0$$

gives 2 real linearly independent sol's,
 $e^{ax} \cos bx, e^{ax} \sin bx$.

$$\begin{aligned} y_1, y_2 \text{ ind:} \\ c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx &\equiv 0 \\ \Rightarrow c_1 \cos bx + c_2 \sin bx &\equiv 0 \\ D_x \Rightarrow -c_1 \sin bx + c_2 b \cos bx &\equiv 0 \\ @ x=0 \text{ get } c_1 + 0 &= 0 \\ 0 + b c_2 &= 0 \\ \text{so } c_1 = c_2 &= 0 \end{aligned}$$

Case III complex roots

Each pair of conjugate roots $a+bi, a-bi$ yield a pair of linearly ind. sol's
 $e^{ax} \cos bx, e^{ax} \sin bx$

If $r = a+bi$ appears with multiplicity > 1 in the characteristic poly, you also get sol's

$$x e^{ax} \cos bx, x e^{ax} \sin bx$$

$\vdots$

following the same algorithm as for repeated real roots as in Case II.

example 1

$$\begin{cases} y'' + 2y' + 5 = 0 \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$$

```
> with(plots):
Warning, the name changecoords has been redefined
> plot(exp(-x)*(cos(2*x)+1.5*sin(2*x)), x=0..4, color=black);
```

$$p(r) = r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$e^{(-1+2i)x} = e^{-x} (\cos 2x + i \sin 2x)$$

$$y(x) = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x$$

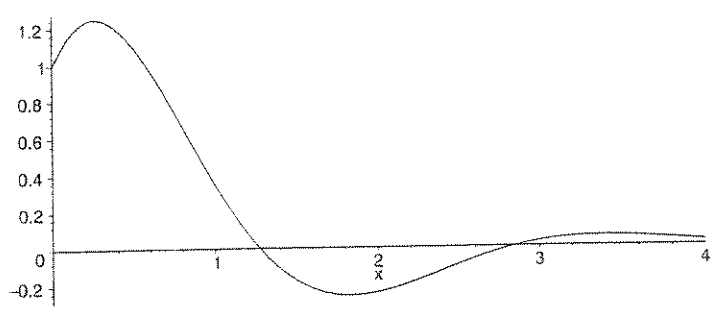
$$y(0) = 1 = c_1$$

$$y'(0) = 2 = -c_1 + 2c_2 \leftarrow \text{why?}$$

$$2 = -1 + 2c_2$$

$$\frac{3}{2} = c_2$$

$$y(x) = e^{-x} \left(\cos 2x + \frac{3}{2} \sin 2x \right)$$



5.4 Mechanical vibrations (= application, so x(t) instead of y(x)).

$$m x'' + c x' + k x = 0$$

$\uparrow$ mass $\uparrow$ coeff of friction $\uparrow$ Hooke's constant.

(see lecture notes 10/22 page 3 for derivation)

example 2

• interpret example 1 in this context.

We study this extremely important DE in stages:

Case 1 Free undamped motion

$\uparrow$ no driving force $\uparrow$ $c=0$

$$m x'' + k x = 0$$

$$x'' + \frac{k}{m} x = 0$$

$$x'' + \omega_0^2 x = 0 \quad (\omega_0 := \sqrt{\frac{k}{m}})$$

if $x = e^{rt}$, $p(r) = r^2 + \omega_0^2 = 0$

$$r = \pm i \omega_0$$

$$e^{i \omega_0 t} = \cos \omega_0 t + i \sin \omega_0 t$$

$$x(t) = A \cos \omega_0 t + B \sin \omega_0 t$$

simple harmonic motion

example 3 A mass of 2 kg oscillates without damping on a spring with Hooke's constant 18 Newtons/meter. It is initially stretched 1 meter from equilibrium, and released with a velocity of $\frac{3}{2}$ m/sec. Show that the displacement $x(t)$ solves

A)
$$\begin{cases} x'' + 9x = 0 \\ x(0) = 1 \\ x'(0) = \frac{3}{2} \end{cases}$$

B) Solve the IVP (A). Express the solution in amplitude-phase form, and interpret the $(t, x(t))$ graph below (Identify amplitude, phase, time delay).

```
> with(plots):
  Digits:=5:
  omega:=3;           #angular frequency
  alpha:=arctan(.5);  #phase
  delta:=arctan(.5)/3; #time delay
  C:=sqrt(5./4);      #amplitude
  T:=evalf(2*Pi/3);   #period

                                0.46365
                                0.15455
                                1.1180
                                T:= 2.0944
  plot1:=plot(cos(3*t)+.5*sin(3*t), t=0..4, color=black):
  plot2:=plot(C*cos(3*t-alpha), t=0..4, color=black):
  plot3:=plot(C*cos(3*(t-delta)), t=0..4, color=black):
  display({plot1, plot2, plot3});
```

