Math 2250-4
Monday March 7

§5.3 homogeneous linear DE's with const. coeff's.

Recall general linear DE set up from Friday page 3, §5.2:

$$\mathcal{L}(y) = y^{(n)} + p_{n-1}y^{(n-1)} + \dots + p_1 y' + p_0 y \qquad p_j(x) \text{ continuous on interval } I \text{ also } f(x).$$

$\mathcal{L}(y) = 0$ homogeneous DE

$\mathcal{L}(y) = f$ inhomogeneous

We know (pages 3-4 Fri):

- IVP $\begin{cases} \mathcal{L}(y) = f \\ y(a) = b_0 \\ y'(a) = b_1 \\ \vdots \\ y^{(n-1)}(a) = b_{n-1} \end{cases}$ has unique solutions
- Solution space to $\mathcal{L}(y) = 0$ is n-dimensional vector space.
- You can use the Wronskian to see if you've found a basis (how?)

And (Tuesday notes, page 3)

- The general solution to $\mathcal{L}(y) = f$ is $y = y_P + y_H$ (y_P: partic soln; y_H: general soln to homog. eqtn.)

Because

(i) If $\mathcal{L}(y_P) = f$ and $\mathcal{L}(y_H) = 0$

then $\mathcal{L}(y_P + y_H) = \mathcal{L}(y_P) + \mathcal{L}(y_H) = f + 0 = f$

so $y_P + y_H$ is always a soltn.

(ii) If $\mathcal{L}(y_P) = f$ and also $\mathcal{L}(y) = f$

then $\mathcal{L}(y - y_P) = f - f = 0$

so $y - y_P = y_H$ (a homog sol'n)

$y = y_P + y_H$.

<u>Example 1</u> On Friday we showed that for

$$\mathcal{L}(y) = y''' - 3y'' - 4y' + 12y$$

the soltns to $\mathcal{L}(y) = 0$ are

$$y_H(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{3x} \qquad (\{e^{2x}, e^{-2x}, e^{3x}\} \text{ are a basis for the sol'n space})$$

book writes $y_c(x)$, for "complementary sol'n".

Find the full solution to

$$y''' - 3y'' - 4y' + 12y = 6$$

hint: try a particular sol'n which is constant.

<u>Example 5</u> Rework the chapter 1 problem

$$y' - 7y = 14$$

to find $y = y_P + y_H$. What is a basis (? dimension) of the sol'n space to $\mathcal{L}(y) = 0$?

to motivate the following theorem, do <u>Example 3</u>, p. 5 Friday.

<u>How to solve constant coefficient linear homogeneous DE's</u>

$$\mathcal{L}(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = 0 \qquad a_j\text{'s } \underline{\text{constant}}$$

<u>step 1</u> try $y = e^{rx}$.

Then $\mathcal{L}(y) = \underbrace{(r^n + a_{n-1}r^{n-1} + \dots + a_1 r + a_0)}_{p(r),\ \text{the characteristic polynomial}} e^{rx}$

So any root of $p(r)$ yields a solution

Now there a several cases of increasing complexity.

<u>Case I</u> If $p(r)$ has n distinct (different) real roots $r_1, r_2, \dots r_n$ then

$$y_H(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}$$

is the general soltn (i.e. $\{e^{r_1 x}, e^{r_2 x}, \dots e^{r_n x}\}$ is a basis)

at $x=0$ the Wronskian matrix is

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & & r_n \\ r_1^2 & r_2^2 & & r_n^2 \\ \vdots & \vdots & & \vdots \\ r_1^{n-1} & r_2^{n-1} & & r_n^{n-1} \end{bmatrix}$$

with det $= \prod_{i<j} (r_j - r_i) \neq 0$!, so these n fcns are lin ind, and basis!

(Example 4 Monday is this case)

<u>Case II</u> Repeated roots (real).

If $(r-\alpha)^k$ is a factor of $p(r)$, then

- $e^{\alpha x}, x e^{\alpha x}, \dots x^{k-1} e^{\alpha x}$ are k linearly ind. soltns.
- If $p(r) = (r-r_1)^{k_1}(r-r_2)^{k_2} \cdots (r-r_j)^{k_j}$ $\quad k_1 + k_2 + \dots + k_j = n$ then a basis of soltns is given by

$$\{\underbrace{e^{r_1 x}, xe^{r_1 x}, \dots x^{k_1 - 1} e^{r_1 x}}_{k_1 \text{ fcns}}, \underbrace{e^{r_2 x}, xe^{r_2 x}, \dots x^{k_2-1} e^{r_2 x}}_{k_2 \text{ fcns}}, \dots \underbrace{e^{r_j x}, xe^{r_j x}, \dots x^{k_j - 1} e^{r_j x}}_{k_j \text{ fcns}}\}$$

one <u>could</u> check that these n functions are linearly independent (harder than case I), and that they all solve $\mathcal{L}(y) = 0$ (page 312).

Example 6 $\quad y'''' - y''' = 0$ ⟵ This is an example of Case II, and it's easy enough that we can check ans by antidifferentiation:

$$p(r) = r^4 - r^3 = r^3(r-1)$$

so $\boxed{y_H(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^x}$

$\uparrow$ $1 = e^{0x}$

Case III $\quad$ Complex roots: exponential fcns still work, you just need to remember (or learn!) Euler's formula.

Recall from Taylor series in Calc.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

If we write $i = \sqrt{-1}$ then define

$$e^{ix} := 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \ldots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \ldots$$

cos x!

i sin x

$$\boxed{e^{ix} := \cos x + i \sin x} \qquad \text{Euler's formula}$$

Define, for $z = x + iy$

$$\boxed{e^z = e^{x+iy} := e^x e^{iy} = e^x(\cos y + i \sin y)}$$

Now, return to the DE $L(y) = 0$, with $r = a \pm ib$ roots of the characteristic polynomial. We claim

$$e^{(a+ib)x}, \quad e^{(a-ib)x} \qquad \text{are (complex) solutions.}$$

$\uparrow$ $(f(x) + ig(x))' := f'(x) + ig'(x)$

$\Rightarrow L(f + ig) = L(f) + iL(g)$ so if $y = f + ig$ is a complex solution, this is equivalent to $L(f) = 0 = L(g)$

This follows from the fact that

$$\boxed{\frac{d}{dx} e^{(a+ib)x} = (a+ib)e^{(a+ib)x}}$$

$\uparrow$

a good exercise in trig identities, p. 315

Thus if $r = a \pm ib$ are conjugate roots of the characteristic poly $p(r)=0$,

then box on left of bottom page 2 says

$$D_x e^{rx} = re^{rx}.$$

Thus $\mathcal{L}(e^{rx}) = p(r)e^{rx}$
$\equiv 0$ since $r = a \pm ib$ are roots.

On the other hand

$$e^{rx} = e^{ax}(\cos bx \pm i \sin bx)$$
$$= f + ig$$

$$\text{so } \underset{0+0i}{\overset{\shortparallel}{0}} = \mathcal{L}(e^{rx}) = \underbrace{\mathcal{L}(e^{ax}\cos bx)}_{\text{real}} \pm i\,\underbrace{\mathcal{L}(e^{ax}\sin bx)}_{\text{imaginary}}.$$

thus $\mathcal{L}(e^{ax}\cos bx) = 0$, $\mathcal{L}(e^{ax}\sin bx) = 0$. (2 sol'ns from two roots, $a \pm ib$).

lin ind: $c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx \equiv 0$

$\div e^{ax} \Rightarrow$ $c_1 \cos bx + c_2 \sin bx \equiv 0$

D_x: $-c_1 b \sin bx + c_2 b \cos bx \equiv 0$

@ $x=0$ yields $c_1 + 0 = 0$
$0 + bc_2 = 0$
$\Rightarrow c_1 = c_2 = 0$.

Example 7 Find the general sol'n to

$$y'' + 4y = 0$$

(notice this is our undamped spring from page 3 of last Wednesday notes, written with different letters).

$p(r) = r^2 + 4 = 0$
$r^2 = -4$
$r = \pm 2i$

$e^{(2i)x} = e^{i(2x)} = \cos 2x + i\sin 2x$

$y_H = c_1 \cos 2x + c_2 \sin 2x$ ✓

Case III completed: If $p(r) = (r-(a+ib))^k q(r)$ then $(r-(a-ib))^k$ is also a factor (since we assume $p(r)$ has real number coeff's)

and $e^{ax}\cos bx$, $e^{ax}\sin bx$
$xe^{ax}\cos bx$, $xe^{ax}\sin bx$
$\vdots$
$x^{k-1}e^{ax}\cos bx$, $x^{k-1}e^{ax}\sin bx$
are $2k$ linearly ind. sol'ns.

combining cases I, II, III yields an algorithm for solving $\mathcal{L}(y) = 0$ whenever $\mathcal{L}$ is an n^{th} order, const-coeff linear DE operator.