Math 2250-4
Friday March 4 §5.2-5.3

finish Wed notes: pages 5, 2
Then:

HW for March 11
5.1 2, 9, 11, 17, 27 29, 30, 31, 34, 35, 39
5.2 2 5, 9, 11, 13 21, 22 25 26
5.3 3, 10, 14 21 22 24 29, 33, 37
5.4 4, 5, 7 10 12, 15, 17, 18, 23
in 15, 17, 18 use Maple (or equiv) for the graphing portion.

Example 1: Find all solutions to the 3rd order linear homogeneous differential equation

$$\mathcal{L}(y) = y'''(x) - 3y''(x) - 4y' + 12y = 0$$

let's try $y = e^{rx}$:
$y' = re^{rx}$
$y'' = r^2 e^{rx}$
$y''' = r^3 e^{rx}$

$$\mathcal{L}(y) = \underbrace{(r^3 - 3r^2 - 4r + 12)}_{\text{want roots of the characteristic polynomial}} e^{rx}$$

possible integer roots divide 12
$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12,$

$r = 2$: $8 - 12 - 8 + 12 = 0$ ✓

$$\begin{array}{r} r^2 - r - 6 \\ r-2 \,\overline{)\, r^3 - 3r^2 - 4r + 12} \\ \underline{r^3 - 2r^2 \qquad\qquad} \\ -r^2 - 4r \qquad \\ \underline{-r^2 + 2r \qquad} \\ -6r + 12 \\ \underline{-6r + 12} \\ 0 \end{array}$$

$p(r) = (r-2)(r^2 - r - 6) = (r-2)(r+2)(r-3)$.

deduce
$y_1 = e^{2x}$
$y_2 = e^{-2x}$
$y_3 = e^{3x}$
are solutions.

$\mathcal{L}$ is a linear operator, (see page 3)

so
$$\begin{aligned} \mathcal{L}(c_1 y_1 + c_2 y_2 + c_3 y_3) &= \mathcal{L}(c_1 y_1) + \mathcal{L}(c_2 y_2) + \mathcal{L}(c_3 y_3) \\ &= c_1 \mathcal{L}(y_1) + c_2 \mathcal{L}(y_2) + c_3 \mathcal{L}(y_3) \\ &= 0 \end{aligned}$$

so $c_1 y_1 + c_2 y_2 + c_3 y_3$ also satisfies this linear homogeneous DE.

In physics you call this the principle of superposition.

[If $\mathcal{L}(y) = f$, $\mathcal{L}(z) = g$ then $\mathcal{L}(y+z) = f + g$]
$\mathcal{L}(ay + bz) = af + bg$
a, b const.

So if V is our solution space to $\mathcal{L}(y) = 0$, we see that any
$c_1 y_1 + c_2 y_2 + c_3 y_3 \in V$.

Is $\text{span}\{y_1, y_2, y_3\}$ all of V?
Are y_1, y_2, y_3 linearly independent?

(continued on next page!)

Example 2 : Solve the initial value problem:

$$\text{IVP}\begin{cases} y'''(x) - 3y''(x) - 4y' + 12y = 0 \\ y(0) = 0 \\ y'(0) = -3 \\ y''(0) = 5 \end{cases}$$

$y_1 = e^{2x}$
$y_2 = e^{-2x}$
$y_3 = e^{3x}$

$y(x) = c_1 y_1 + c_2 y_2 + c_3 y_3$
$y' = c_1 y_1' + c_2 y_2' + c_3 y_3'$
$y'' = c_1 y_1'' + c_2 y_2'' + c_3 y_3''$

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

If you have n solutions to a linear homogeneous DE, the matrix for which column j is $\begin{bmatrix} y_j \\ y_j' \\ \vdots \\ y_j^{n-1} \end{bmatrix}$ is called the Wronskian matrix. Its determinant is called the Wronskian $W(y_1, y_2, y_n)$

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} e^{2x} & e^{-2x} & e^{3x} \\ 2e^{2x} & -2e^{-2x} & 3e^{3x} \\ 4e^{2x} & 4e^{-2x} & 9e^{3x} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

at $x=0$ this is the matrix eqn.

$$\begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & 3 \\ 4 & 4 & 9 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Could we have solved any IVP for this DE, at $x=0$? Could the Wronskian explain why?

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> with(LinearAlgebra) :
  A := Matrix(3, 4, [1, 1, 1, 0,
                     2, -2, 3, -3,
                     4, 4, 9, 5]);
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$$A := \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & -2 & 3 & -3 \\ 4 & 4 & 9 & 5 \end{bmatrix}$$

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> ReducedRowEchelonForm(A);
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$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

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>
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Since $W(y_1, y_2, y_3) \neq 0$, every IVP (at $x=0$) solution is a unique linear combination of y_1, y_2, y_3.

See theory pages, theorems 2 & 3. $\rightarrow$ In particular, they span the solution space and are linearly independent

So they are a basis

$$\boxed{y(x) = -2e^{2x} + e^{-2x} + e^{3x}}$$

Theory pages

Recall from Wed, (did $n=2$)
an n^{th} order differential operator

$$\mathcal{L}(y) = P_n(x)\, y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \dots + P_1(x)\, y'(x) + P_0(x)\, y(x)$$

is called linear, because

$$* \begin{cases} \mathcal{L}(y_1 + y_2) = \mathcal{L}(y_1) + \mathcal{L}(y_2) \\ \mathcal{L}(c y_1) = c\,\mathcal{L}(y_1) \end{cases}$$

The differential equation

$$\mathcal{L}(y) = F(x)$$

is homogeneous if $F(x) \equiv 0$. Otherwise it is inhomogeneous

Theorem 1 : The solution space to the linear, homogeneous, DE

$$\mathcal{L}(y) = 0$$

is a vector space, i.e. it is closed under addition and scalar multiplication.
Thus if $y_1, \dots y_n$ are solutions, so is

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n .$$

why : If y_1, y_2 are solutions, then $\mathcal{L}(y_1 + y_2) \overset{\text{by } *}{=} \mathcal{L}(y_1) + \mathcal{L}(y_2) = 0 + 0$
so $y_1 + y_2$ is.

If y_1 is a solution, then
$\mathcal{L}(c y_1) = c\,\mathcal{L}(y_1) = 0$, so $c y_1$ is too.

More generally,

$$\mathcal{L}(c_1 y_1 + c_2 y_2 + \dots + c_n y_n) = \mathcal{L}(c_1 y_1) + \mathcal{L}(c_2 y_2 + \dots + c_n y_n)$$
$$= c_1 \underbrace{\mathcal{L}(y_1)}_{0} + \underbrace{\mathcal{L}(c_2 y_2)}_{0} + \mathcal{L}(c_3 y_3 + \dots + c_n y_n)$$
$$= \dots = 0 + 0 + \dots + 0 = 0.$$

Theorem 2 : Let

$$\text{IVP} \begin{cases} \mathcal{L}(y) = y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \dots + p_0(x)\, y \equiv 0 \\ y(a) = b_0 \\ y'(a) = b_1 \\ \vdots \\ y^{(n-1)}(a) = b_{n-1} \end{cases}$$

If $p_{n-1}, \dots p_0$ are continuous on an interval I
and if $a \in I$.
Then there is always exactly 1 solution to this initial value problem.

why : makes intuitive sense.
more explanation later in course.

Theorem 3 : Let $\mathcal{L}(y) = y^{(n)} + p_{n-1}y^{(n-1)} + \dots + p_1 y' + p_0 y$ be as in theorem 2. Then the solution space to $\mathcal{L}(y)=0$ has <u>dimension n</u>. Thus if we can find n linearly independent solutions they will span the solution space.

<u>why</u>: Using the existence-uniqueness result, Theorem 2, we can find (well, actually, there exist) $y_1 \dots y_n$ solving $\mathcal{L}(y_j)=0$, with initial values

$$\begin{bmatrix} y_1(a) \\ y_1'(a) \\ \vdots \\ y^{(n-1)}(a) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \begin{bmatrix} y_2(a) \\ y_2'(a) \\ \vdots \\ y_2^{(n-1)}(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots \begin{bmatrix} y_n(a) \\ y_n'(a) \\ \vdots \\ y_n^{(n-1)}(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

$\{y_1, \dots y_n\}$ are a basis:

(a) span: let $y(x)$ solve $\mathcal{L}(y)=0$. Then

$y(a)=b_0$
$y'(a)=b_1$
$\vdots$
$y^{(n-1)}=b_{n-1}$
some #'s $b_0, \dots b_{n-1}$

compare $y(x)$ to $b_0 y_1(x) + b_1 y_2(x) + \dots + b_{n-1} y_{n-1}(x)$

these 2 solutions solve the same IVP, so they must be the same function, by Theorem 2. i.e. $y(x) = b_0 y_1 + b_1 y_1 + \dots + b_{n-1} y_{n-1}$

(b) linearly ind: if $c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$
then
$c_1 y_1' + c_2 y_2' + \dots + c_n y_n' = 0$
$\vdots$
$c_1 y_n^{(n-1)} + \dots + c_n y_n^{(n-1)} = 0$

at $x=a$ deduce
$c_1 = 0$
$c_2 = 0$
$\vdots$
$c_n = 0$ ∎

Theorem 4 : Let $\mathcal{L}(y)$ be as in Theorems 2 & 3.
Let $\{y_1, \dots y_n\}$ be n solutions to $\mathcal{L}(y)=0$. on the interval I.

Let $W = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{bmatrix}$ (the Wronskian).

Let $a \in I$. Then if $W(a) \neq 0$, $y_1, \dots y_n$ are a basis!
(so you don't need to compute W at every x!)

<u>why</u>: $W(a) \neq 0$ means every IVP at a has a unique solution expressed as a linear combination of $y_1, y_2, \dots y_n$.

Example 3 Find the solution space to

$$y'' - 6y' + 9 = 0$$

try $y = e^{rx}$
$y' = re^{rx}$
$y'' = r^2 e^{rx}$

$\mathcal{L}(y) = (r^2 - 6r + 9)e^{rx} = 0$.

$(r-3)^2 = 0$

$y_1 = e^{3x}$.

Where's y_2?

trick: try $y_2 = xe^{3x}$
$y_2' = (1+3x)e^{3x}$ (product rule)
$y_2'' = (3 + 3(1+3x))e^{3x}$

$$\mathcal{L}(y_2) = e^{3x}\begin{bmatrix} x(9-18+9) \\ +1(6-6) \end{bmatrix} = 0!$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}(1+3x) \end{bmatrix}$$

$W(0) = \det \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = 1$ so $\{y_1, y_2\}$ is a basis (see Theorem 4 p 4)

$$\boxed{y(x) = c_1 e^{3x} + c_2 xe^{3x}}$$

(note, $W(x) = e^{9x}$ is never 0).

example 3 is typical of double roots. if $p(r) = (r-\alpha)^2 = r^2 - 2\alpha + \alpha^2$, for the 2^{nd} order, linear, homogeneous, const-coeff DE

$$y'' - 2\alpha y' + \alpha^2 y = 0$$

then $y_1 = e^{\alpha x}$, $y_2 = xe^{\alpha x}$ are a basis for the solution space

note
$y_2 = xe^{\alpha x}$ $\cdot\,\alpha^2$
$y_2' = e^{\alpha x}(\alpha x + 1)$ $\cdot\,-2\alpha$
$y_2'' = e^{\alpha x}(\alpha + \alpha(\alpha x + 1))$ $\cdot\,1$

$$\mathcal{L}(y_2) = e^{\alpha x}\begin{bmatrix} x(\alpha^2 - 2\alpha^2 + \alpha^2) \\ +1(-2\alpha + 2\alpha) \end{bmatrix} = 0.$$