Math 2250-4
Monday 3/28
§10.2-10.3 Laplace transform magic cont'd.

Continue filling in Laplace transform table and solving initial value problems.

warmup exercise

$\mathcal{L}\{7 - 5\cos 3t + 2\sin 8t\}(s)$

$\mathcal{L}^{-1}\left\{\frac{2}{s+3} + \frac{4}{s^2+9}\right\}(t)$

table exercises

3a $\mathcal{L}\{e^{at} f(t)\}(s)$

1b use $\cosh(kt) = \frac{1}{2}(e^{kt} + e^{-kt})$

$\sinh(kt) = \frac{1}{2}(e^{kt} - e^{-kt})$

2a We checked table entry for $f'(t)$, $f''(t)$ before break.
Check $f'''(t) = (f''(t))'$ to see pattern.

	$f(t)$	$F(s) := \int_0^\infty e^{-st} f(t)\,dt,\ s>M$	already checked
	$f(t)+g(t)$	$F(s)+G(s)$	✓
	$c f(t)$	$c F(s)$	✓
0	1	$1/s$	✓
0	t	$1/s^2$	
0	t^n	$n!/s^{n+1}$ $n \in \mathbb{N}$	
0	$e^{\alpha t}$	$1/(s-\alpha)$	✓
1	$\cos kt$	$s/(s^2+k^2)$	✓
1	$\sin kt$	$k/(s^2+k^2)$	✓
1	$\cosh(kt)$	$s/(s^2-k^2)$	
1	$\sinh(kt)$	$k/(s^2-k^2)$	
2	$f'(t)$	$sF(s) - f(0)$	✓
2	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	✓
2	$f'''(t)$	$s^3F(s) - s^2f(0) - sf'(0) - f''(0)$	
2	etc.		
2	$\int_0^t f(\tau)\,d\tau$	$F(s)/s$	
2 analogous!	$t f(t)$	$-F'(s)$	
2	$t^2 f(t)$	$F''(s)$	
2	$t^3 f(t)$	$-F'''(s)$	
2	etc.		
2	$f(t)/t$	$\int_s^\infty F(\sigma)\,d\sigma$	
3 analogous!	$e^{at} f(t)$	$F(s-a)$	
3	$u(t-a)$	e^{-as}/s	
3	$u(t-a) f(t-a)$	$e^{-as} F(s)$	
	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$	✓
	$e^{at}\sin kt$	$k/((s-a)^2+k^2)$	✓
4	$t\cos kt$	$(s^2-k^2)/(s^2+k^2)^2$	
4	$\frac{1}{2k} t \sin kt$	$s/(s^2+k^2)^2$	
4	$\frac{1}{2k^3}(\sin kt - kt\cos kt)$	$1/(s^2+k^2)^2$	
4	$t e^{at}$	$1/(s-a)^2$	
	& more!		

(2b) let $g(t) = \int_0^t f(\tau)\,d\tau$

note $g'(t) = f(t)$
$g(0) = 0$

so $\mathcal{L}\{g'(t)\}(s) = s\,G(s) - g(0)$
$F(s) = s\,G(s)$
$\frac{F(s)}{s} = G(s)$

(2c)
$$F'(s) = \lim_{\Delta s \to 0} \frac{1}{\Delta s}\left(F(s+\Delta s) - F(s)\right)$$
$$= \lim_{\Delta s \to 0} \underbrace{\frac{1}{\Delta s}\left(\int_0^\infty f(t)e^{-(s+\Delta s)t}\,dt - \int_0^\infty f(t)e^{-st}\,dt\right)}_{\int_0^\infty f(t)\left(\frac{e^{-(s+\Delta s)t} - e^{-st}}{\Delta s}\right)dt}$$
$$\downarrow$$
$$\frac{d}{ds}e^{-st} = -t\,e^{-st}$$
$$= \int_0^\infty -t\,f(t)e^{-st}\,dt = \mathcal{L}\{-tf(t)\}(s)$$

so $\mathcal{L}\{t\,f(t)\}(s) = -F'(s)$
$\mathcal{L}\{t\cdot t f(t)\}(s) = -(-F'(s))' = F''(s)$ etc.

powers of t (0), (4)

$\mathcal{L}\{1\}(s) = 1/s$ (before break)
$\mathcal{L}\{t\cdot 1\}(s) = -D_s(s^{-1}) = s^{-2}$
$\mathcal{L}\{t\cdot t\}(s) = -D_s(s^{-2}) = 2s^{-3}$
$\mathcal{L}\{t\cdot t^2\}(s) = -D_s(2s^{-3}) = 3!\,s^{-4}$ etc.

$\mathcal{L}\{t e^{\alpha t}\}(s) = -D_s(s-\alpha)^{-1} = \frac{1}{(s-\alpha)^2}$ ← so if $\alpha = a$ is real,
$\mathcal{L}\{t e^{at}\}(s) = \frac{1}{(s-a)^2}$

if $\alpha = ik$
$$\mathcal{L}\{t e^{ikt}\}(s) = \frac{1}{(s-ik)^2}\,\frac{(s+ik)^2}{(s+ik)^2} = \frac{(s^2-k^2) + 2iks}{(s^2+k^2)^2}$$
so $\mathcal{L}\{t\cos kt\}(s) + i\,\boxed{\mathcal{L}\{t\sin kt\}(s)} = \frac{s^2-k^2}{(s^2+k^2)^2} + i\,\boxed{\left(\frac{2ks}{(s^2+k^2)^2}\right)}$

note $\frac{s^2-k^2}{(s^2+k^2)^2} - \frac{s^2+k^2}{(s^2+k^2)^2} = \frac{-2k^2}{(s^2+k^2)^2}$

so $\left(t\cos kt - \frac{1}{k}\sin kt\right) = -2k^2\,\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}(t)$

Exercise 1   Use partial fractions and table entries to find

$f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$ so

$$F(s) = \frac{s^2 + 3s + 1}{s^3 - 4s}$$

Exercise 2   Resonance revisited! (p. 606)

Solve the IVP

$$\begin{cases} x'' + \omega_0^2 x = F_0 \sin \omega t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

$\mathcal{L}$: $\quad s^2 X(s) + \omega_0^2 X(s) = F_0 \dfrac{\omega}{s^2+\omega^2}$

$$X(s)(s^2+\omega_0^2) = F_0 \frac{\omega}{s^2+\omega^2}$$

$\omega \neq \omega_0$

$$X(s) = F_0 \omega \frac{1}{(s^2+\omega^2)(s^2+\omega_0^2)}$$

$$= F_0 \omega \left[\frac{1}{s^2+\omega^2} - \frac{1}{s^2+\omega_0^2} \right] \frac{1}{\omega_0^2 - \omega^2}$$

so

$$x(t) = \frac{F_0 \omega}{\omega_0^2 - \omega^2} \left[\frac{1}{\omega} \sin \omega t - \frac{1}{\omega_0} \sin \omega_0 t \right]$$

much quicker than Chapter 5 techniques!

$\omega = \omega_0$

$$X(s) = F_0 \omega_0 \frac{1}{(s^2+\omega_0^2)^2}$$

$$x(t) = F_0 \omega \frac{1}{2\omega_0^3} (\sin \omega_0 t - \omega_0 t \cos \omega_0 t)$$

table!

resonance.

Exercise 3 Solve

$$\begin{cases} y^{(4)} + 2y'' + y = 4te^t \\ y(0)=0 \\ y'(0)=0 \\ y''(0)=0 \\ y'''(0)=0 \end{cases}$$

$\mathcal{L}$:

$$s^4 Y(s) - s^3\cdot 0 - s^2\cdot 0 - s\cdot 0 - 0 + 2s^2 Y(s) - s\cdot 0 - 0 + Y(s) = 4\,\frac{1}{(s-1)^2}$$

$$Y(s)\,(\underbrace{s^4+2s^2+1}_{(s^2+1)^2}) = 4\,\frac{1}{(s-1)^2}$$

$$Y(s) = \frac{4}{(s^2+1)^2(s-1)^2}$$

partial fractions bonanza!
(can you set it up correctly?)

> with(inttrans); #integral transform package

[addtable, fourier, fouriercos, fouriersin, hankel, hilbert, invfourier, invhilbert, invlaplace, invmellin, laplace, mellin, savetable]

exercise 1

> $F := s \to \frac{(s^2+3\cdot s+1)}{(s^3-4\cdot s)}$;

$$F := s \to \frac{s^2+3s+1}{s^3-4s}$$

> convert(F(s), parfrac, s); #check partial fractions

$$-\frac{1}{4s} - \frac{1}{8(s+2)} + \frac{11}{8(s-2)}$$

> invlaplace(F(s), s, t);

$$\frac{5}{4}\cosh(2t) + \frac{3}{2}\sinh(2t) - \frac{1}{4} = -\frac{1}{4} - \frac{1}{8}e^{-2t} + \frac{11}{8}e^{2t}$$

exercise 3

> $Y := s \to \frac{4}{(s^2+1)^2\cdot(s-1)^2}$;
convert(Y(s), parfrac, s);
invlaplace(Y(s), s, t);

$$Y := s \to \frac{4}{(s^2+1)^2(s-1)^2}$$

$$\frac{1}{(s-1)^2} - \frac{2}{s-1} + \frac{1+2s}{s^2+1} + \frac{2s}{(s^2+1)^2}$$

$$2\cos(t) + e^t(-2+t) + \sin(t)(1+t)$$

>