Math 2250-4 ①
Wed 3/2
§5.1 2^{nd} order linear DE's
p. 1-2 theory } we'll discuss in parallel
p. 3-4 examples }
First: discuss page 3 Tuesday, then §5.1

Def. A second order linear differential equation has the form
§5.1
$$A(x)y'' + B(x)y' + C(x)y = F(x)$$
We search for solutions $y(x)$ on some interval $[a,b]$.
In this chapter we assume $A(x) \neq 0$ on the interval, so the DE could also be written (after dividing by $A(x)$)
$$y'' + p(x)y' + q(x)y = f(x)$$
On reason this differential eqtn is called linear is that the "operator" $\mathcal{L}$ defined by
$$\mathcal{L}(y) = y'' + p(x)y' + q(x)y$$
satisfies the linear operator axioms (like matrix mult axioms)
(1) $\mathcal{L}(y_1 + y_2) = \mathcal{L}(y_1) + \mathcal{L}(y_2)$
(2) $\mathcal{L}(cy_1) = c\mathcal{L}(y_1)$
[we checked these in detail for a particular example on Monday].
Therefore, by vector space theory, the solution set to the homogeneous DE
$$\mathcal{L}(y) = y'' + py' + qy = 0$$
is a subspace. And the general solution to $\mathcal{L}(y) = f$ is $y = y_P + y_H$, where y_P is any particular solution, and y_H is the general soln to the homogeneous equation.

Existence-Uniqueness theorem
If $p(x), q(x)$ are continuous on the interval I, and $a \in I$. Then there is a unique soltn to

IVP $\begin{cases} y''(x) + p(x)y' + q(x)y = f(x) \\ y(a) = b_0 \\ y'(a) = b_1 \end{cases}$

$\boxed{\text{In theory, text uses } x = \text{variable},\ y(x) = \text{function}}$

one reason this makes sense:
$$y'' = f - py' - qy$$
so if I know $y(a), y'(a)$, then I know $y''(a)$.
But
$$y''' = f' - p'y' - py'' - q'y - qy'$$
so I can find $y'''(a)$.
so I can find all derivs of y at a, i.e. the Taylor series for y.
[of course this would require all fcns to be infinitely differentiable.]

Consequence of the existence uniqueness theorem:

<u>Theorem</u> The solution space to the homogeneous DE

$\mathcal{L}(y) = y'' + py' + qy = 0$

is <u>2-dimensional</u>. (So our goal is to find a basis consisting of <u>2 linearly ind. solutions</u>)

<u>proof</u>: let $y_1(x)$ solve $\mathcal{L}(y_1)=0$, $\begin{cases} y_1(a)=1 \\ y_1'(a)=0 \end{cases}$

$y_2(x)$ solve $\mathcal{L}(y_2)=0$ $\begin{cases} y_2(a)=0 \\ y_2'(a)=1. \end{cases}$

These solutions exist by existence - uniqueness theorem, and are unique

then the unique solution to

$$\begin{cases} y''+py'+qy=0 \\ y(a)=b_0 \\ y'(a)=b_1 \end{cases}$$

is $y(x) = b_0 y_1(x) + b_1 y_2(x)$

is a soltn ✓

$y(a) = b_0 \cdot 1 + b_1 \cdot 0 = b_0$ ✓

$y'(a) = b_0 \cdot 0 + b_1 \cdot 1 = b_1$ ✓.

thus $\{ y_1(x), y_2(x) \}$ are a basis for the solution space.

(they span by above.

they are independent because

$c_1 y_1 + c_2 y_2 \equiv 0$

$\Rightarrow c_1 y_1' + c_2 y_2' \equiv 0$

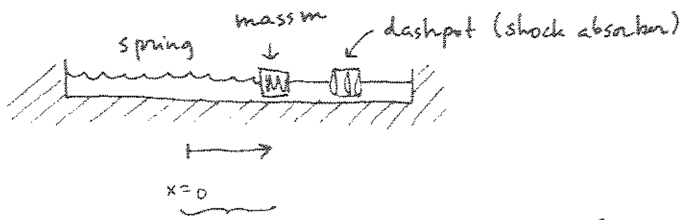
at $x=a$ we get

$c_1 + 0 = 0$

$0 + c_2 = 0$ so $c_1 = c_2 = 0$).

$\blacksquare$

Our most important example in the entire course



$x(t)$ = displacement of mass from equilibrium.

Model

Newton says

$$m\,x''(t) = \text{net force}$$

$$m\,x''(t) = F_S + F_D + F(t)$$

spring force — drag from dashpot — any other applied forces.

Good models for spring and drag forces:

If $F_S = F_S(x)$ [only depends on displacement]

then $F_S(x) = F_S(0) + F_S'(0)\,x + \frac{1}{2}F_S''(0)\,x^2 + \dots$ (Taylor series).

- for small x ignore quadratic & above terms
- $F_S(0) = 0$ since $\begin{matrix} x=0 \\ x'=0 \end{matrix}$ is equilibrium.

so $F_S \approx F_S'(0)\,x = -k\,x$ (coeff should be <0 since $x>0 \Rightarrow F_S<0$, $x<0 \Rightarrow F_S>0$)

Hooke's "Law"

If $F_D = F_D(v)$, where $v = x'(t)$

then $F_D(v) \approx \underset{\overset{\shortparallel}{0}}{F_D(0)} + F_D'(0)\,v + \cancel{\dots}$

$F_D(v) \approx -c\,v$ linear drag

so $m\,x''(t) = -kx - cv + F(t)$ is a good model **sometimes**

$$\boxed{\text{IVP} \begin{cases} m\,x''(t) + c\,x'(t) + kx = F(t) \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}}$$

$$\boxed{\begin{array}{l} \text{In applications} \\ \text{text uses} \\ \quad t = \text{variable} \\ \quad x(t) = \text{function} \end{array}}$$

<u>example</u> In the mass-spring system suppose $m = 2$, $k = 8$, $c = 0$, $F(t) = 0$

$$\underset{\text{kg}}{m = 2}, \quad \underset{\text{N/m}}{k = 8}$$

$$2x''(t) + 8x'(t) = 0$$

$$x'' + 4x = 0$$

$x_1(t) = \cos 2t$, $x_2(t) = \sin 2t$ are solutions, check!

they are linearly ind., since if

$$c_1 x_1 + c_2 x_2 \equiv 0$$

then

$$c_1 x_1' + c_2 x_2' \equiv 0$$

$$\begin{bmatrix} x_1 & x_2 \\ x_1' & x_2' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

at $t = 0$: $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = c_2 = 0.$

this also shows we can solve any IVP:

$$\begin{cases} x'' + 4x = 0 \\ x(0) = 1 \\ x'(0) = -2 \end{cases}$$

$$c_1 x_1(0) + c_2 x_2(0) = 1$$
$$c_1 x_1'(0) + c_2 x_2'(0) = -2$$

$$c_1 + 0 = 1$$
$$0 + 2c_2 = -2$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 2 & -2 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

$c_1 = 1$
$c_2 = -1$

$$\boxed{x(t) = \cos 2t - \sin 2t}$$

Can you write $x(t)$ is phase-amplitude form?

how to find a basis for the sol'n space to

$$ay'' + by' + cy = 0$$

where $a \neq 0, b, c$ are constants.

try $y = e^{rx}$
$y' = re^{rx}$
$y'' = r^2 e^{rx}$

$$\Rightarrow \mathcal{L}(y) = \underbrace{(ar^2 + br + c)}_{\text{set this poly in } r = 0.} e^{rx}$$

called characteristic polynomial

example

$$y'' - 5y' + 6y = 0$$

if $y = e^{rx}$
get $(r^2 - 5r + 6)\cancel{e^{rx}} = 0$

$(r-3)(r-2) = 0$

$r = 2, 3$

$y_1 = e^{2x}$
$y_2 = e^{3x}$

$$\boxed{y(x) = c_1 e^{2x} + c_2 e^{3x}}$$

$\rightarrow$ lin ind? $c_1 y_1 + c_2 y_2 \equiv 0$
$c_1 y_1' + c_2 y_2' \equiv 0$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \equiv 0$$

$$\begin{bmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{bmatrix}$$

at $x = 0$ $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$; det $= 3 - 2 = 1$
so $c_1 = c_2 = 0$

In general,

$$W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$ is called the Wronskian

if it is $\neq 0$ anywhere, solutions are l.i.

in our example

$$W = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x} \neq 0 \text{ for all } x.$$

Conclusion:
$\{e^{2x}, e^{3x}\}$ is a basis for the 2-dim'l soltn space