Math 2250-4
Friday March 18

Today we jump to chapter 10, Laplace transform. It ties in beautifully with chapter 5.
(we cover 6, 7, 9 after 10.)

HW for next Wed March 30
10.1 3 7 8, 13, 20, 21, 23, 28
10.2 3, 4 5 6, 14 19, 20, 28 31
10.3 3, 7, 8, 17, 20 27, 31

Exam 2 is Thurs. March 31!
covers thru 10.3

Laplace transforms

Let $f(t)$ be a function defined for $t \geq 0$ s.t. f grows at most exponentially fast ($|f(t)| \leq Ce^{Mt}$ for some C, M)
Then the Laplace transform function $F(s) = \mathcal{L}\{f(t)\}(s)$ is defined for $s > M$ by

$$\underset{F(s)}{\underset{\|}{\mathcal{L}\{f(t)\}(s)}} := \int_0^\infty e^{-st} f(t)\,dt$$

examples

- $\mathcal{L}\{1\}(s) = \int_0^\infty e^{-st}\, 1\, dt = \frac{e^{-st}}{-s}\Big]_0^\infty = \frac{1}{s}$ $\quad (s>0)$
- $\mathcal{L}\{e^{\alpha t}\}(s) = \int_0^\infty e^{-st} e^{\alpha t} dt = \int_0^\infty e^{(\alpha - s)t} dt = \frac{1}{\alpha - s} e^{(\alpha-s)t}\Big]_0^\infty = \frac{1}{s-\alpha}$ $\quad s>\alpha$, or $s > Re(\alpha)$ if α is complex.
- Laplace transform is linear!

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\}(s) = \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t))\,dt$$
$$= c_1 \int_0^\infty e^{-st} f_1(t)\,dt + c_2 \int_0^\infty e^{-st} f_2(t)\,dt$$
$$= c_1 F_1(s) + c_2 F_2(s)$$

- $\alpha = ik$ imaginary

$$\Rightarrow \mathcal{L}\{e^{ikt}\}(s) = \frac{1}{s-ik}\cdot\frac{s+ik}{s+ik} = \frac{s+ik}{s^2+k^2} = \frac{s}{s^2+k^2} + i\boxed{\frac{k}{s^2+k^2}}$$

|| Euler

$$\mathcal{L}\{\cos kt + i \sin kt\}(s) = \mathcal{L}\{\cos kt\}(s) + i\boxed{\mathcal{L}\{\sin kt\}(s)}$$

- similarly, if $\alpha = a + ik$ then

$$\mathcal{L}\{e^{at}\cos kt + i e^{at}\sin kt\}(s) = \mathcal{L}\{e^{\alpha t}\}(s) = \frac{1}{s-\alpha} = \frac{1}{s-(a+ik)}$$
$$= \frac{1}{(s-a)-ik}\cdot\frac{(s-a)+ik}{(s-a)+ik}$$

$$\mathcal{L}\{e^{at}\cos kt\}(s) + i\boxed{\mathcal{L}\{e^{at}\sin kt\}(s)} = \frac{s-a}{(s-a)^2+k^2} + i\boxed{\frac{k}{(s-a)^2+k^2}}$$

Exercise 1

Compute $\mathcal{L}\{7 - 5e^{2t} + 4\cos 2t\}(s)$

$f(t)$	$F(s)$
1	$1/s$
e^{at}	$\frac{1}{s-a}$
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$\vdots$	
to be continued!	

Exercise 2 It's a (difficult to prove) theorem that a given Laplace transform $F(s)$ can arise from at most one continuous function $f(t)$. So, you can use linearity and a good table to compute inverse Laplace transforms.

Find $\mathcal{L}^{-1}\left\{\frac{4}{s^2+1} - \frac{s}{s^2+4}\right\}(t)$

Laplace transforms and DE's :

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty \underbrace{e^{-st}}_{v}\,\underbrace{f'(t)\,dt}_{du} = e^{-st}f(t)\Big]_{t=0}^{\infty} - \int_0^\infty -se^{-st}f(t)\,dt$$

$dv = -se^{-st}dt$

$u = f(t)$

$$= -f(0) + sF(s) \qquad (s>M)$$

(see book cover)

$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$\vdots$	

so

$$\mathcal{L}\{f''(t)\}(s) = s\underbrace{\mathcal{L}\{f'(t)\}(s)}_{(-f(0)+sF(s))} - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

Exercise 3 : Solve IVP with Laplace !! (without x_p, x_H, etc.).

$$* \begin{cases} x'' + 4x(t) = 10\cos 3t \\ x(0) = 2 \\ x'(0) = 1 \end{cases}$$

(non-resonant undamped forced oscillator)

Answer: write $\mathcal{L}\{x(t)\}(s) = X(s)$. We know a unique sol'n $x(t)$ to $*$ exists. for this sol'n,

$$\mathcal{L}\{x''(t)\}(s) + 4\mathcal{L}\{x(t)\}(s) = 10\,\mathcal{L}\{\cos 3t\}(s)$$

$$s^2X(s) - s(2) - 1 + 4X(s) = 10\cdot\frac{s}{s^2+9}$$

$$X(s)(s^2+4) = \frac{10s}{s^2+9} + 2s + 1$$

$$X(s) = 10s\left(\frac{1}{(s^2+4)(s^2+9)}\right) + \frac{2s+1}{s^2+4}$$

$$= 10s\left(\frac{1}{s^2+4} - \frac{1}{s^2+9}\right)\frac{1}{5} + \frac{2s+1}{s^2+4}$$

$$= \frac{4s}{s^2+4} + \frac{1}{s^2+4} - \frac{2s}{s^2+9}$$

so,

$$\boxed{x(t) = 4\cos 2t + \tfrac{1}{2}\sin 2t - 2\cos 3t}$$

Magic!!

Exercise 4   Solve an unforced damped oscillator problem

$$\begin{cases} x'' + 6x' + 34x = 0 \\ x(0) = 3 \\ x'(0) = 1 \end{cases}$$

via Laplace transform.

Laplace:

$$s^2 X(s) - 3s - 1 + 6(sX(s) - 3) + 34X(s) = 0$$

$$X(s)(s^2 + 6s + 34) = 3s + 1 + 18 = 3s + 19$$

$$X(s) = \frac{3s+19}{s^2+6s+34}$$ ← to use table, complete square in denom, then complete the linear in numerator

$$= \frac{3s+19}{(s+3)^2 + (34-9)}$$

$$= \frac{3s+19}{(s+3)^2 + 25}$$

$$X(s) = \frac{3(s+3)}{(s+3)^2+25} + \frac{10}{(s+3)^2+25}$$

So $\boxed{x(t) = 3e^{-3t}\cos 5t + 2e^{-3t}\sin 5t}$!

in §10.1-10.3 we fill in more of the Laplace transform table entries, solve applied and general IVP's, get very good at partial fractions and at completing the square, and even see some new applications

well, these are mostly in 10.4-10.5, which we'll cover in later HW's.