§5.6 Damped forced oscillations $(c>0)$

$$mx'' + cx' + kx = F_0 \cos\omega t$$

$$\begin{aligned}
k\,(\quad & x_p = A\cos\omega t + B\sin\omega t \\
+c\,(\quad & x_p' = -A\omega\sin\omega t + B\omega\cos\omega t \\
+m\,(\quad & x_p'' = -A\omega^2\cos\omega t - B\omega^2\sin\omega t
\end{aligned}$$

$$\begin{aligned}
L(x_p) = \ & \cos\omega t\,[kA + Bc\omega - mA\omega^2] \quad = \cos\omega t\,(F_0) \\
& + \sin\omega t\,[kB - Ac\omega - mB\omega^2]
\end{aligned}$$

$$\begin{bmatrix} k-m\omega^2 & c\omega \\ -c\omega & k-m\omega^2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

so $$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{(k-m\omega^2)^2 + c^2\omega^2}\begin{bmatrix} k-m\omega^2 & -c\omega \\ c\omega & k-m\omega^2 \end{bmatrix}\begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{(k-m\omega^2)^2 + c^2\omega^2}\begin{bmatrix} F_0(k-m\omega^2) \\ F_0 c\omega \end{bmatrix}}$$

So $A\cos\omega t + B\sin\omega t$

$$= C\cos(\omega t - \alpha)$$

$$C = \frac{F_0\sqrt{(k-m\omega^2)^2 + c^2\omega^2}}{(k-m\omega^2)^2 + c^2\omega^2} = \frac{F_0}{\sqrt{\underbrace{(k-m\omega^2)^2}_{\text{small if } \omega \approx \omega_0} + \underbrace{c^2\omega^2}_{\text{small if } c \approx 0}}}$$

$$x_H(t) = \begin{cases} c_1 e^{r_1 t} + c_2 e^{r_2 t} & r_1 < r_2 < 0 \text{ underdamped.} \\ e^{-pt}(c_1 + c_2 t) & \text{critically damped} \\ e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t) & \text{underdamped.} \end{cases}$$

So, full soltn

$$x(t) = \underbrace{x_p(t)}_{\text{periodic}} + \underbrace{x_H(t)}_{\substack{\text{decays exponentially to zero} \\ \text{regardless of choices of } c_1, c_2 \text{ (or } A, B)}}$$

So we call $x_p(t) = C\cos(\omega t - \alpha) = x_{sp}(t)$; steady periodic

$x_H(t) = x_{tr}(t)$; transient sol'n, reflects initial values *example*

$$\begin{cases} x'' + 2x' + 26x = 82\cos 4t \\ x(0) = 6 \\ x'(0) = 0 \end{cases}$$

could redo page 2 with numbers rather than letters (exam possibility), but we can also plug in:

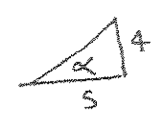
$$x_p(t) = A\cos 4t + B\sin 4t$$

$m = 1$
$\omega = 4$
$c = 2$
$k = 26$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{100+64}\begin{bmatrix} 82\cdot 10 \\ 82\cdot 8 \end{bmatrix} = \begin{bmatrix} \frac{820}{164} \\ \frac{656}{164} \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$k - m\omega^2 = 26 - 16 = 10$
$c\omega = 8$
$F_0 = 82$

$$x_p(t) = 5\cos 4t + 4\sin 4t = \sqrt{41}\cos(4t - \alpha)$$



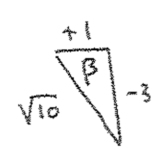
$\alpha = \arctan(.8)$.

$x_H(t)$:

$$r^2 + 2r + 26 = 0$$
$$(r+1)^2 + 25 = 0$$
$$(r + 1 + 5i)(r + 1 - 5i) = 0$$
$$r = -1 \pm 5i$$
$$x_H(t) = e^{-t}(A\cos 5t + B\sin 5t)$$

$$x(t) = 5\cos 4t + 4\sin 4t + e^{-t}(A\cos 5t + B\sin 5t).$$

$$\left.\begin{array}{l} x(0) = 6 = 5 + A \Rightarrow A = 1 \\ x'(0) = 0 = 0 + 16 - A + 5B = 15 + 5B \Rightarrow B = -3 \end{array}\right\}$$



$$x(t) = \sqrt{41}\cos(4t - \alpha) + e^{-t}\sqrt{10}\cos(5t - \beta)$$

$\alpha = \arctan(.8)$ $\quad$ $\beta = \arctan(-3)$

$x_{sp}(t)$ $\quad$ $x_{tr}(t)$.

Practical resonance (see page)

$$x_{sp}(t) = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + c^2\omega^2}} \cos(\omega t - \alpha)$$

if c is small and $\omega^2 \approx \omega_0^2 = \frac{k}{m}$ $\Big\}$ denom tiny $\Rightarrow$ amplitude of x_{sp} huge (relative to F_0)

This is called <u>practical resonance</u>

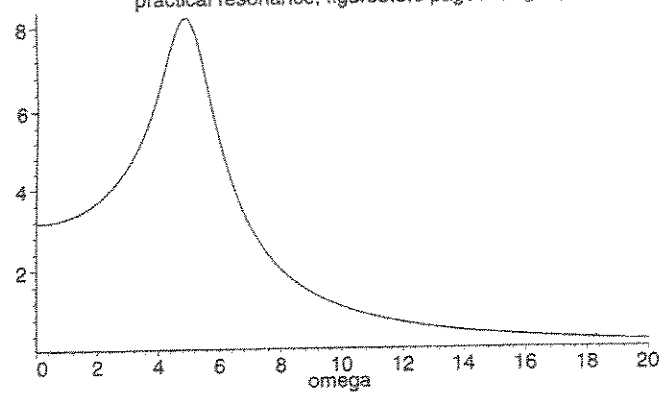
Vary ω in page 3 example

$$x'' + cx' + 26x = 82\cos\omega t$$

$$C = \frac{82}{\sqrt{(26-\omega^2)^2 + c^2\omega^2}} = C(\omega)$$

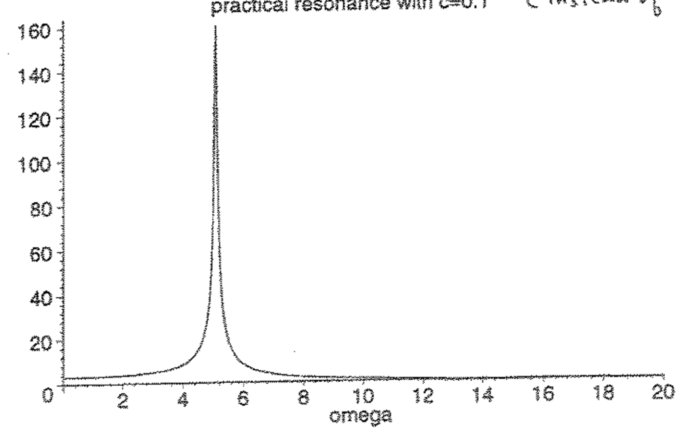
```
> plot(82/sqrt((26-omega^2)^2+4*omega^2),omega=0..20,
  color=black, title='practical resonance, figure5.6.9 page361');
```

practical resonance, figure5.6.9 page 361




```
> plot(82/sqrt((26-omega^2)^2+.01*omega^2),omega=0..20,
  color=black, title='practical resonance with c=0.1');
```

practical resonance with c=0.1 (instead of 4)

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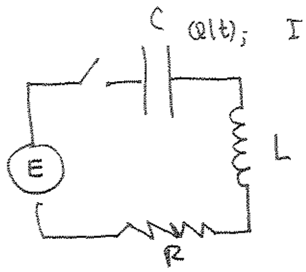
the mechanical ~ electrical analogy con't.



$Q(t)$; $I = Q'(t)$

$$L I' + R I + \frac{Q}{C} = E = E_0 \sin \omega t$$

so

$$L I'' + R I' + \frac{1}{C} I = E_0 \omega \cos \omega t$$

transcribe from page 1

$$I(t) = I_{sp}(t) + I_{tr}(t)$$

$$I_{sp}(t) = I_0 \cos(\omega t - \alpha)$$

amplitude

for $I_0 = \dfrac{E_0 \omega}{\sqrt{\left(\frac{1}{C} - L\omega^2\right)^2 + R^2 \omega^2}}$

$$I_0 = \frac{\omega}{\omega} \frac{E_0}{\sqrt{\left(\frac{1}{\omega C} - L\omega\right)^2 + R^2}}$$

the denominator $Z = \sqrt{\left(\frac{1}{\omega C} - L\omega\right)^2 + R^2}$ is called the <u>impedance</u> of the circuit.

maximum response for fixed E_0, R, when $L\omega = \dfrac{1}{\omega C}$, $C = \dfrac{1}{L\omega^2}$ if C adjustable

$L = \dfrac{1}{C\omega^2}$ if L adjustable.

old radions

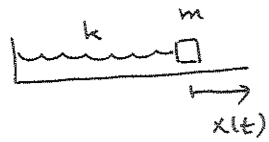
old-style demo!

practical resonance.

If you're an engineer concerned about resonance or practical resonance, you need to know how to figure out natural frequencies of mechanical systems.

The easiest way is often to use conservation of energy (essentially the antiderivative of Newton's law).

- We've done (via Newton)



mass - spring

Energy way:

KE + PE $\equiv$ const.

$\frac{1}{2}m(x'(t))^2 + \frac{1}{2}kx^2 \equiv$ const

$\downarrow$

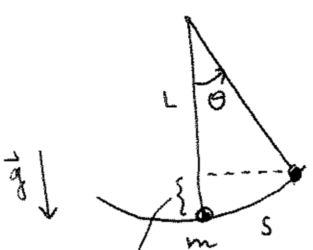
$W = \int_0^x F(x)dx = \int_0^x ks\,ds = \frac{1}{2}ks^2\Big]_0^x = \frac{1}{2}kx^2$

take D_t: $\frac{1}{2}m\,2x'x'' + \frac{1}{2}k\,2xx' \equiv 0$

$x'[\underbrace{mx''+kx}_{\text{the DE!}}] \equiv 0$

$\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$

- We've done (via KE+PE)



$h = L(1-\cos\theta)$

KE + PE $\equiv$ const

$\frac{1}{2}m\,s'(t)^2 + mg\,h \equiv$ const

$\frac{1}{2}m(L\theta')^2 + mg\,L(1-\cos\theta) \equiv$ const

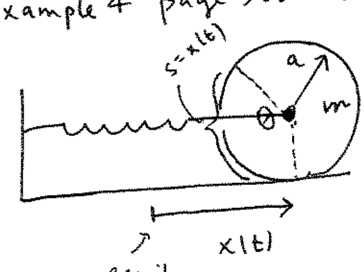
D_t: $mL^2\theta'\theta'' + mg\,L(\sin\theta(t))\theta'(t) \equiv 0$

$Lm\,\theta'(t)\,[L\theta'' + g\sin\theta] \equiv 0$

θ small $\rightarrow$ $L\theta'' + g\theta \cong 0$

$\theta'' + \frac{g}{L}\theta = 0$ $\boxed{\omega_0 = \sqrt{\frac{g}{L}}}$

- Example 4 page 353 (see also HW)



rolling wheel.

$KE = \underbrace{\frac{1}{2}m(x')^2}_{\text{from motion of center of mass}} + \underbrace{\frac{1}{2}I\omega^2}_{\text{rotational KE}}$; $PE = \frac{1}{2}kx^2$

$\omega = \frac{d\theta}{dt}$

I = moment of inertia $= \frac{ma^2}{2}$ for disks

$x = a\theta$

D_t: $x' = a\omega$; $\omega = \frac{x'}{a}$

$KE + PE = \frac{1}{2}m(x')^2 + \underbrace{\frac{1}{2}\,\frac{1}{2}ma^2\left(\frac{x'}{a}\right)^2}_{\frac{1}{4}m(x')^2} + \frac{1}{2}kx^2 = \frac{3}{4}m(x')^2 + \frac{1}{2}kx^2 \equiv$ const

D_t: $\frac{3}{2}mx'x'' + kxx' \equiv 0$

$x'[\frac{3}{2}mx'' + kx] \equiv 0$

$\boxed{\omega_0 = \sqrt{\frac{2k}{3m}}}$