Math 2250-4
Tues. March 15

- Variation of parameters method of finding y_p, page 5 Mon. notes) maybe went on this

then begin §5.6 : forced oscillations and resonance

Overview

$$m x'' + c x' + k x = F_0 \cos \omega t$$

- undamped $(c=0)$

$$m x'' + k x = F_0 \cos \omega t$$

- $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$ $\quad x_p = A\cos\omega t + B\sin\omega t$ $\quad = C\cos(\omega t - \alpha)$

  general soltn $x = x_p + x_H$ $= C\cos(\omega t-\alpha) + C_0\cos(\omega_0 t - \alpha_0)$

- $\omega = \omega_0$ $\quad x_p = t(A\cos\omega_0 t + B\sin\omega_0 t)$ $\quad = t\, C\cos(\omega_0 t - \alpha)$ $\quad$ general soltn $x = x_p + x_H$

  RESONANCE!

- $\omega \neq \omega_0$, but ω close to ω_0
  BEATING

- damped $(c>0)$ (More details Monday)

$$x_p = A\cos\omega t + B\sin\omega t$$
$$= C\cos(\omega t - \alpha)$$

$x_H \rightarrow$ 3 possibilities (over, critically, underdamped)
key feature they all share is $x_H(t) \rightarrow 0$ exponentially fast as $t \rightarrow \infty$.

so x_p is called x_{sp} (steady periodic)
x_H is called x_{tr} (transient)

when damping c is small, and ω is near ω_0, then amplitude C of x_{sp} will (often) be a large multiple of forcing amplitude F_0. This is called practical resonance (or, if you deal in electronics, you might call it resonance.)

Example 1 p.354: $m=1,\ k=9,\ F_0=80,\ \omega=5,\ \omega_0=?$

$$\begin{cases} x''(t) + 9x(t) = 80\cos 5t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

x_H:

x_p:

ans

$x(t) = 5\cos 3t - 5\cos 5t$

period $T_0 = \frac{2\pi}{3}$ (for $5\cos 3t$), period $T_1 = \frac{2\pi}{5}$ (for $5\cos 5t$)

(and all integer multiples).

the period of the sum will be the least common integer multiple of T_0 and T_1, which in this case is

1b) $\boxed{}$

1c) for $x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$

$x(t) = x_p + x_H$.

When will the sum be periodic, when won't it be?

Example 1 p. 354

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> with(plots):
> plot(5*cos(3*t)-5*cos(5*t),t=0..6*Pi,color=black,
  title='Figure 5.6.2 page 350, period=?');
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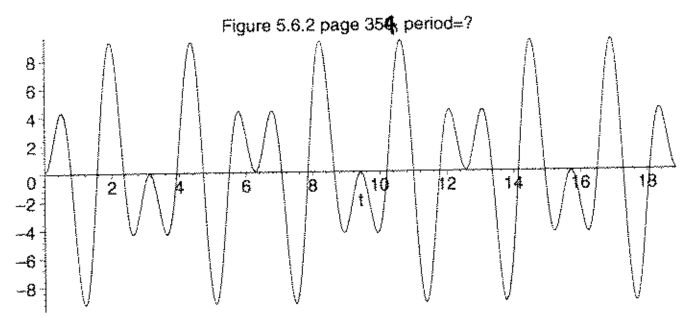


undamped forced IVP, $\omega \neq \omega_0$, with letters

$$\begin{cases} x'' + \frac{k}{m} x = \frac{F_0}{m} \cos\omega t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$$\begin{array}{ll} +\frac{k}{m} & (x_p = A\cos\omega t) \\ +0 & (x_p' = -A\omega\sin\omega t) \\ +1 & (x_p'' = -A\omega^2\cos\omega t) \\ \hline \end{array}$$

$$L(x_p) = \cos\omega t \; A\left[\underbrace{\tfrac{k}{m}}_{\omega_0^2} - \omega^2\right]$$

deduce $A(\omega_0^2 - \omega^2) = \frac{F_0}{m}$

$A = \frac{F_0}{m}(\omega_0^2 - \omega^2)$

so,

$x_p(t) = -\frac{F_0}{m}\frac{1}{(\omega^2-\omega_0^2)}\cos\omega t$. Note $x_H(t) = A\cos\omega_0 t + B\sin\omega_0 t$.

so, by plugging in or observation
IVP solution is

$$\boxed{x(t) = \frac{+F_0}{m(\omega^2-\omega_0^2)}\left(\cos\omega_0 t - \cos\omega t\right) + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

check IVP!

when ω is near (but $\neq 0$) ω_0,
this sum varies between ± 2,
depending on whether the two
terms are in, or out of phase....
trig makes this precise!!

$$\cos(\alpha-\beta) - \cos(\alpha+\beta)$$
$$\|$$
$$\cos\alpha\cancel{\cos\beta} + \sin\alpha\sin\beta - (\cancel{\cos\alpha}\cos\beta - \sin\alpha\sin\beta)$$
$$= 2\sin\alpha\sin\beta$$

let $\left.\begin{array}{l}\alpha = \bar{\omega}t = \left(\frac{\omega+\omega_0}{2}\right)t \\ \beta = \delta t = \left(\frac{\omega-\omega_0}{2}\right)t\end{array}\right\}$ so $\begin{array}{l}\alpha-\beta = \omega_0 t \\ \alpha+\beta = \omega t\end{array}$

i.e.

$$\boxed{x(t) = \frac{F_0}{m(\omega^2-\omega_0^2)} \cdot 2\sin\bar{\omega}t\,\sin\delta t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

period $\frac{2\pi}{\bar{\omega}}$, $\bar{\omega} = \omega+\omega_0$

period $\frac{2\pi}{\delta}$ is huge if $\omega \approx \omega_0$; $\delta = \frac{\omega-\omega_0}{2}$

Beating

Notice the beating amplitude $\frac{2F_0}{m(\omega^2-\omega_0^2)}$ blows up as $\omega \to \omega_0$

Example 2 (not the text's).

Keep the same data as in #1 : $m=1$, $k=9$, $F_0=80$

$\omega_0=3$

except choose $\omega=3.1$

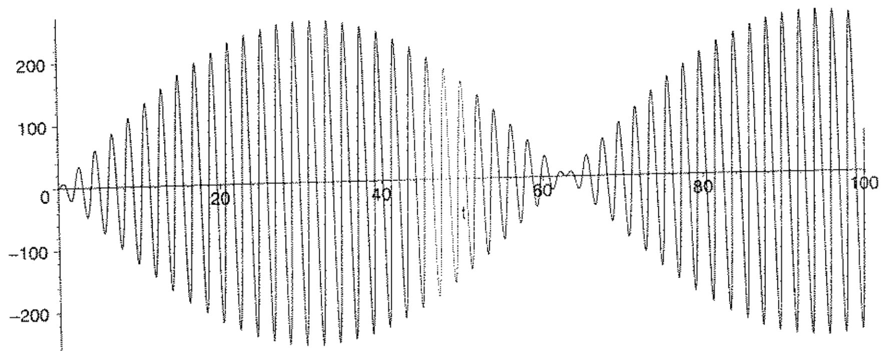
Use second box to solve IVP

$$\begin{cases} x''+9x=80\cos(3.1t) \\ x(0)=0 \\ v(0)=0 \end{cases}$$

and compute the beating period and amplitude

ans $x(t) \cong 262.3 \sin(3.05t)\sin(.05t)$

$T_{beat} \approx 126$ (actually ≈ 63)



Resonance! $\omega = \omega_0$

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

using §5.5, guess

$$\begin{aligned} +\omega_0^2 \, (\quad & x_p = t\,(A\cos\omega_0 t + B\sin\omega_0 t) \quad) \\ 0 \, (\quad & x_p' = t\,(-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t) + A\cos\omega_0 t + B\sin\omega_0 t \quad) \\ +1 \, (\quad & x_p'' = t\,(-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t) + \left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right]2 \quad) \end{aligned}$$

$$L(x_p) = t\,(0) + 2\left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega_0 t$$

Deduce $A = 0$, $B = \frac{F_0}{2m\omega_0}$

$$\underbrace{x_p(t) = \frac{F_0}{2m\omega_0}\, t\sin\omega_0 t}_{\text{sats } x(0)=0,\ x'(0)=0}$$

, so IVP soln is

$$\boxed{x(t) = \frac{F_0}{2m\omega_0}\, t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{x_0}{\omega_0}\sin\omega_0 t}$$

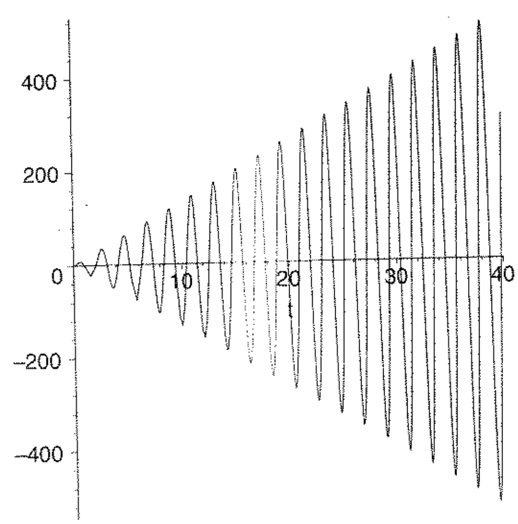
(you can also guess this by letting $\omega \to \omega_0$ on page 3, second box, via linearization!)

No matter how small $F_0 \neq 0$, soln blowup as $t \to \infty$!

Example 3, as before $m = 1$, $k = 9$ $(\omega_0 = 3)$
$F_0 = 80$
$\omega = 3 = \omega_0$

$$\begin{cases} x'' + 9x = 80\cos 3t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

$x(t) = ?$



ans $x(t) = \frac{40}{3}\, t\sin 3t$