So much for unforced (homogeneous) spring problems. Now discuss general non-homogeneous linear DE, then in §5.6 we'll return to springs:

Math 2250-4
Monday March 14

§5.5 : Finding y_p's for $\mathcal{L}(y)=f$.

Recall, if $\mathcal{L}$ is a linear operator the general sol'n to

$$\mathcal{L}(y)=f$$

is $\quad y = y_p + y_H$

where y_p is any particular sol'n and
y_H is the general (n-dim'l) solution to $\mathcal{L}(y)=0$

How to find y_p's.

Book calls the method "undetermined coefficients", for constant coefficient linear DE's, is really just "guessing" (but $\underbrace{\text{could be}}_{\text{is!}}$ justified with vector space theory).

example 1

$$\mathcal{L}(y) = y'' + 3y' + 4y$$

solve $\mathcal{L}(y) = 3x+2$

try $y_p = Ax + B$ (because when you apply $\mathcal{L}$ to polys of degree ≤ 1 you get back polys of degree ≤ 1).

$$\begin{aligned} 4(\quad & y_p = Ax+B \\ +\,3(\quad & y_p' = A \\ +\,1(\quad & y_p'' = 0 \end{aligned}$$

$$\mathcal{L}(y_p) = x(4A) + 1(3A+4B) = 3x+2$$

$$\left.\begin{aligned} 4A &= 3 \\ 3A+4B &= 2 \end{aligned}\right\} \quad \begin{aligned} A &= 3/4 \\ \tfrac{9}{4} + 4B &= 2 \\ 4B &= -\tfrac{1}{4} \\ B &= -\tfrac{1}{16} \end{aligned}$$

$$\boxed{y_p = \frac{3}{4}x - \frac{1}{16}}$$

y_H: $\quad r^2 + 3r + 4 = 0$

$$r = \frac{-3 \pm \sqrt{9-16}}{2} = -\frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

$$\boxed{y_H(x) = e^{-3/2\,x}\left(A\cos\tfrac{\sqrt{7}}{2}x + B\sin\tfrac{\sqrt{7}}{2}x\right)}$$

$$\boxed{y(x) = \frac{3}{4}x - \frac{1}{16} + e^{-3/2\,x}\left(A\cos\tfrac{\sqrt{7}}{2}x + B\sin\tfrac{\sqrt{7}}{2}x\right)}$$

<u>example</u> 2 $y'' - 4y = 2e^{3x}$

find y_p:

try $y_p = Ae^{3x}$

$L(e^{3x}) = 9e^{3x} - 4e^{3x} = 5e^{3x}$

so $L(Ae^{3x}) = 5Ae^{3x}$

$\qquad\qquad\qquad\quad \| $

$\qquad\qquad\qquad\quad 2$; $\quad A = \frac{2}{5} \qquad y_p = \frac{2}{5}e^{3x}$

<u>example</u> 3 $y'' - 4y = 10e^{2x}$

find y_p

try

oh oh. → the RHS was related to y_H. There's a way to fix your guess:

<u>example</u> 4 find y_p to $y'' - 4y = 4e^{3x} + 5e^{2x}$

<u>ans</u>: (hint: look at the 2 examples above)

guessing rules

If $L(y)$ is constant coeff. linear differential operator, what would be your first guess at the form of y_p if you wanted to solve the following, and why?

$L(y) = 3\cos 2x$

$y_p =$

$L(y) = 4e^{2x}\sin 3x$

$y_p =$

$L(y) = x^3 + 6x^2 - 5$

$y_p =$

$L(y) = 7e^{13x}$

$y_p =$

$L(y) = x\sin 2x$

$y_p =$

Remark: The linear algebra reason that "guessing" works: If V is finite-dim'l and $L: V \to V$ has nullspace $= \{0\}$ (only sol'n to $Lv = 0$ is $v = 0$), then $Lv = f$ has a unique sol'n v_p.

there are special rules if $f(x) = \text{RHS}$ is related to y_H: you multiply your guess for y_p by x^s, where s is the lowest natural power making each term in your guess NOT a solution of the homog. eq'n (s is the power of $r-\alpha$ in the characteristic polynomial, $p(r) = (r-\alpha)^s q(r)$.).

See table page 346, & rule 2 page 345

more examples

$L(y) = 7e^{-2x} + 3e^{-2x}\sin x$

$L = D^2 + 4D + 5I \quad (L(y) = y'' + 4y' + 5y)$

$p(r) = r^2 + 4r + 5 = (r+2+i)(r+2-i)$

$y_p = y_{p_1} + y_{p_2} =$

$L(y) = 7e^{-2x} + 3e^{-2x}\sin x$

$L(y) = y'' + 4y' + 4y$

$p(r) = r^2 + 4r + 4 = (r+2)^2$

$y_p = y_{p_1} + y_{p_2} =$

Maple check (well, actual solutions rather than form of guess for y_p)

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> with(DEtools):
> deqtn1 := diff(y(x), x, x) + 4·diff(y(x), x) + 5·y(x) = 7·exp(-2·x) + 3·exp(-2·x)·sin(x);
  dsolve(deqtn1, y(x));
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$$deqtn1 := \frac{d^2}{dx^2}y(x) + 4\left(\frac{d}{dx}y(x)\right) + 5y(x) = 7e^{-2x} + 3e^{-2x}\sin(x)$$

$$y(x) = e^{-2x}\sin(x)\_C2 + e^{-2x}\cos(x)\_C1 - \frac{1}{2}e^{-2x}(-14 + 3\cos(x)x)$$

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> deqtn2 := diff(y(x), x, x) + 4·diff(y(x), x) + 4·y(x) = 7·exp(-2·x) + 3·exp(-2·x)·sin(x);
  dsolve(deqtn2, y(x));
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$$deqtn2 := \frac{d^2}{dx^2}y(x) + 4\left(\frac{d}{dx}y(x)\right) + 4y(x) = 7e^{-2x} + 3e^{-2x}\sin(x)$$

$$y(x) = e^{-2x}\_C2 + e^{-2x}x\_C1 - \frac{1}{2}(-7x^2 + 6\sin(x))e^{-2x}$$

If guessing won't work (maybe your linear DE is <u>not</u> const coeff) there is a method, <u>variation of parameters</u>, that will.

illustrate method on 2^{nd} example page 2:

$$y''-4y = 10e^{2x}$$

[by improved "guessing" we got

$$y_p = \frac{5}{2}xe^{2x}$$
$$y = \frac{5}{2}xe^{2x} + c_1e^{2x} + c_2e^{-2x}].$$

<u>Variation of parameters</u> : to find y_p if you have a basis $\{y_1, y_2\}$ for y_H

$$\mathcal{L}(y) = y'' + p(x)y' + q(x)y$$

$y_H = \text{span}\{y_1, y_2\}$ known.

to solve

$$\mathcal{L}(y) = f$$

try $y_p = u_1y_1 + u_2y_2$ $\quad u_1, u_2 = $ fcns of x

$$\begin{array}{ll} q(x)\,(& y_p = u_1y_1 + u_2y_2 \\ p(x)\,(& y_p' = u_1y_1' + u_2y_2' + \underbrace{u_1'y_1 + u_2'y_2}_{\text{set } \equiv 0} \\ 1\,(& \Rightarrow y_p'' = u_1y_1'' + u_2y_2'' + \underbrace{u_1'y_1' + u_2'y_2'}_{\text{set} = f} \end{array}$$

so $\mathcal{L}(y_p) = u_1\underbrace{\mathcal{L}(y_1)}_{0} + u_2\underbrace{\mathcal{L}(y_2)}_{0.} + f$

$= f$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}\begin{bmatrix} 0 \\ f \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W}\begin{bmatrix} -y_2f \\ y_1f \end{bmatrix}$$

integrate to get u_1 & u_2 then get y_p.

$y_1 = e^{2x}$

$y_2 = e^{-2x}$

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4$$

$f = 10e^{2x}$

$$u_1' = -\frac{1}{4}\left(-e^{-2x}\,10e^{2x}\right) = \frac{5}{2}$$
$$u_2' = -\frac{1}{4}\left(e^{2x}\,10e^{2x}\right) = -\frac{5}{2}e^{4x}$$

take $u_1 = \frac{5}{2}x$

$u_2 = -\frac{5}{8}e^{4x}$

$$y_p = u_1y_1 + u_2y_2$$
$$= \frac{5}{2}xe^{2x} - \frac{5}{8}e^{4x}e^{-2x}$$

$\boxed{y_p = \frac{5}{2}xe^{2x} - \frac{5}{8}e^{2x}}$, (agrees since $-\frac{5}{8}e^{2x}$ is a y_H)