Math 2250-4
Tuesday 3/1

§ 4.7

- Review general facts about finding basis for vector space (and the ideas behind these facts), page 5 Monday notes. (Good review of yesterday's techniques)
- And now for something completely different, yet completely the same (§ 4.7)

Exercise 1 Recall the vector space $\mathcal{F} = \{$ continuous functions $f: \mathbb{R} \to \mathbb{R}\}$
let $L(y) = y'''$ for $y \in V = \{ f: \mathbb{R} \to \mathbb{R}$ s.t. f, f', f'', f''' are continuous$\}$.
let $W = \{ y(x) \in V$ s.t. $L(y) = 0 \}$

the zero function.

1a) Verify $(\alpha), (\beta)$, so W is a subspace of V

α) $y_1, y_2 \in W \Rightarrow y_1 + y_2 \in W$:

β) $y \in W$, $k \in \mathbb{R} \Rightarrow ky \in W$:

1b) Find all solt'ns $y(x)$ to $y''' = 0$ (by antidifferentiation).
Show W is 3-dim'l
Exhibit a basis for W.

Exercise 2 Linear algebra explains why partial fractions always works! (§4.7)

For example, to solve

$$\frac{3x+4}{(x-2)(x+2)(x-1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x-1}$$

$$= \frac{A(x+2)(x-1) + B(x-2)(x-1) + C(x-2)(x+2)}{(x-2)(x+2)(x-1)}$$

we set numerators equal:

$$3x+4 = A(x+2)(x-1) + B(x-2)(x-1) + C(x-2)(x+2)$$

unique soln (A,B,C) exists iff

$\{(x+2)(x-1), (x-2)(x-1), (x-2)(x+2)\}$ are a basis for P_2

by previous page, just need to check independence, since $\dim P_2 = 3$.

but if $c_1(x+2)(x-1) + c_2(x-2)(x-1) + c_3(x-2)(x+2) \equiv 0$

(means LHS = 0 for all x);

@ $x=1 \Rightarrow c_3 = 0$
@ $x=-2 \Rightarrow c_2 = 0$
@ $x=2 \Rightarrow c_1 = 0$

$\Rightarrow$ independent
$\Rightarrow$ basis by 4 on page 3.

Thus unique A, B, C will exist.
- this can be generalized to explain why all the partial fraction "magic" always works!

4a) Find A, B, C, by plugging in good x-values

4b) Find A, B, C, by "collecting coefficients". (This amounts to expressing equations using the basis $\{1, x, x^2\}$ for P_2)

Connection between exercise 1 example (linear DE's), and matrix equations:

<u>Definition</u> Let V_1 and V_2 be vector spaces.
A function $\mathcal{L}: V_1 \to V_2$ is called <u>linear</u> iff
a) $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g) \quad \forall f, g \in V_1$
b) $\mathcal{L}(kf) = k\,\mathcal{L}(f) \quad \forall f \in V_1,\ k \in \mathbb{R}.$

<u>Example</u> $T(\vec{x}) := A_{m\times n}\vec{x}$
$T: \mathbb{R}^n \to \mathbb{R}^m$
T is linear, since $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$
$A(k\vec{x}) = kA\vec{x}$

<u>Example</u> $L(y) = y'''$
$L: V \to \mathcal{F}$ (page 1)
L is linear, since $(y_1+y_2)''' = y_1''' + y_2'''$
$(ky)''' = ky'''.$

<u>Theorem</u> Let $\mathcal{L}: V_1 \to V_2$ be linear
Let $b \in V_2$
Let $y_P \in V_1$ with $\mathcal{L}(y_P) = b$.
Then
(1) if $\mathcal{L}(y_H) = 0$, then $y = y_P + y_H$ also solves $\mathcal{L}(y) = b$.
(2) if y is <u>any</u> soltn to $\mathcal{L}(y) = b$ then $y = y_P + y_H$ for some $y_H \in V_1$ with $\mathcal{L}(y_H) = 0$

y_P: "particular" soltn
y_H: "homogeneous" soltn.

proof:
(1) if $\mathcal{L}(y_H) = 0$ then $\mathcal{L}(y_P + y_H) = \mathcal{L}(y_P) + \mathcal{L}(y_H) = b + 0 = b.$
(2) if $\mathcal{L}(y) = b$ then $y = y_P + (y - y_P)$
$b = \mathcal{L}(y) = \mathcal{L}(y_P) + \mathcal{L}(y - y_P) = b + \mathcal{L}(y - y_P)$, so $\mathcal{L}(y - y_P) = 0$ ∎

<u>Exercise 3</u> (follow up exercise 1)

(a) Find all soltns to
$$y''' = e^{2x}$$

(b) Find all sol'ns to
$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$