Math 2280−1/2250−4

Newton's law of cooling application − Maple Homework 2

Notes and assignment

Friday January 28 − due Monday February 7 at 5:00 p.m.

Our second Maple homework and first real project is from section 1.5, First Order Linear Differential Equations, specifically from *1.5 Application: Indoor Temperature Oscillations*, pages 58−60. I recommend you work on this project with your text open, carefully following and checking the text's exposition and mathematical computations. If you haven't done so already, first take some time to download, open, excute, and understand the 2250/2280 Maple commands in the file

http://www.math.utah.edu/~korevaar/2250spring11/maplecommands.mw

since every command you need to complete this assignment should be in that list. Of course, you are also encouraged to use the Maple help windows.

Open this Maple assignment from the URL

http://www.math.utah.edu/~korevaar/2250spring11/maplehw2.mw

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Project Directions: You will be handing in a single Maple document containing the answers to the bolded, underlined exercises below. You may work individually, or in pairs. If you work with another person, hand in only one document for the two of you. At the top of this document you should have a text field with an appropriate title, the date, your name(s) and uid(s). Format this information in a pleasing way. Below the title/name/date header you should answer the **exercises** in order. Please delete the part of this top matter that is not relevant to answering the project questions.

The exercises which call for hand work (e.g. #1), can be done by leaving enough text space in your document so that you can write in the relevant explanations/derivations by hand, or by stapeling pages with this work to your Maple printout. Alternately, you may create text displays of your "hand" computations, as is done in this assignment document, by switching between "text" and "math" format in your text fields, using the buttons at the upper left of the Maple window. To get the Math to display nicely, choose the "2−D Math notation" option for input and output display, at the menu item tools/options/display. You probably want this to apply globally, whenever you use Maple.

Note: For clarity of exposition, please only do mathematical computations inside explict Math fields created with the "[>" math menu prompt, like this one:

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To remind you of this formatting requirement, I've already inserted prompts in appropriate places. You may end up inserting additional ones as well. Finally, Use Maple 12 if you're working on the Mathematics Department system (because our higher numbered Maple versions will crash if try 3−d plotting).

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Here's the project set−up as explained in the text, page 58: You have no air conditioner or evaporative cooler for your home, it's summer, you live in the authors' home town of Athens Georgia, and the outside temperatures are oscillating periodically, with a period of 24 hours. In other words, we will assume the ambient temperature outside is given by

$$A(t) = a_0 + a_1 \cos(\omega t) + b_1 \sin(\omega t)$$

where the angular frequency ω is 2π radians per 24 hours, i.e.

$$\omega = \frac{\pi}{12}$$

radians per hour. In the book example the 24−hour average temperature is 80 degrees F, with a minimum of 70 degrees at 4 a.m., and a maximum of 90 degrees at 4 p.m. This is consistent with the fact that the hottest part of the day usually occurs in the afternoon, and the coolest part is in the early morning hours. Using this information, we may write

$$A(t) = 80 - 10\cos(\omega\,(t-4)) \qquad (1)$$

and expand using the cosine addition angle formula, which you should have memorized and which is in your Calculus book:

$$\cos(\alpha+\beta) = \cos(\alpha)\,\cos(\beta) - \sin(\alpha)\,\sin(\beta)$$

This yields

$$A(t) = 80 - 5\cos(\omega\,t) - 5\sqrt{3}\,\sin(\omega\,t)\ . \qquad (2)$$

(Check!! but don't hand in.)

We wish to model the temperature inside the house, assuming no sources or sinks for heat exist inside, and using Newton's Law of Cooling. This model yields the linear DE

$$\frac{du}{dt} = k\,(A(t) - u)$$

for the indoor temperature u(t), i.e.

$$\frac{du}{dt} + k\,u = k\left(a_0 + a_1\cos(\omega\,t) + b_1\sin(\omega\,t)\right)\ . \qquad (3)$$

The proportionality constant k reflects how quickly heat from the outside affects temperature change inside − the smaller k is, the more slowly the inside temperature changes for fixed difference $A(t) - u$, i.e. small k means good insulation.

Exercise 1. Using the integration factor algorithm we've learned for solving first order linear DEs, and working by hand, show that the general solution to the DE (3) above is given just as the book claims in equation 4, page 59. In other words, that the solutions are

$$u(t) = a_0 + c_0\,\mathrm{e}^{-k\,t} + c_1\cos(\omega\,t) + d_1\sin(\omega\,t)\ , \qquad (4)$$

and that the constants c_0, c_1, d_1 satisfy the formulas shown just below equation (4). You may wish to use the book's integral tables, specifically #49,50 on the back cover of our text, as you work this problem by hand.

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Exercise 2. Redo the computation for Exercise 1 using Maple to compute all the integrals you need. Your work here should all be done in the Maple document. A particularly quick way to get the answer is to use the definite integral way of completing the integrating factor algorithm for first order linear initial value problems. In other words, Consider the linear DE initial value problem

$$\frac{dy}{dt} + P(t)y(t) = Q(t)$$

$$y(0) = y_0.$$

Using the equivalent integrating factor formulation for the differential equation that we're already familiar with yields

$$\frac{d}{dt}\left(e^{\int P(t)\,dt}\,y(t)\right) = e^{\int P(t)\,dt}\,Q(t).$$

Choose your anti−derivative $\int P(t)\,dt$ so that its value at $t=0$ is zero, then integrate the expression above from from 0 to t. Use the Fundamental Theorem of Calculus on the left, and a dummy variable other than t inside the integral on the right, because you're already using t as your upper limit, and you get the formula

$$e^{\int P(t)\,dt}\,y(t) - y_0 = \int_0^t e^{\int P(r)\,dr}\,Q(r)\,dr.$$

You can quickly solve that formula for $y(t)$, by adding y_0 to both sides and dividing by the exponential integrating factor.

The Maple commands and output should be shown, along with any text explanation or command comments you find necessary. Maple isn't likely to group terms exactly as you and the text have in equation (4), so explain why Maple's answer and yours agree.

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Notice that the exponential part of the solution to (4) decays to zero (exponentially) as $t \to \infty$, and that what remains in the limit is also a solution (i.e. when the integration constant c_0=0). We call this the steady periodic (sp) solution, and write

$$u_{sp}(t) = a_0 + c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Exercise 3. Use Maple to crunch the numbers and show that with the Athens Georgia temperature data (i.e. the ambient temperature A(t) given by equation (2), and insulation constant

$$k = 0.2$$

we get solutions to the initial value problem $u(0) = u_0$ given by

$$u(t) = 80 + e^{-0.2\,t}\left(u_0 - 82.3351\right) + 2.3351\cos\left(\frac{\pi t}{12}\right) - 5.6036\sin\left(\frac{\pi t}{12}\right) \qquad (5)$$

and so also, in case the constant c_0 in equation (4) above equals zero (i.e. u_0=82.3351), the particular steady periodic solution (hinted at in the dfield plot in our class notes):

$$u_{sp}(t) = 80 + 2.3351\cos\left(\frac{\pi t}{12}\right) - 5.6036\sin\left(\frac{\pi t}{12}\right). \qquad (6)$$

Notice that all solutions converge to the "stable" steady periodic solution as $t \to$To avoid super−long decimals like I did in the displays above, you might want to set

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> Digits := 5 : #the number of digits Maple will use in decimal calculations
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Exercise 4. Keeping $k = 0.2$ and setting $u(0) = u0$, use the "dsolve" command to reproduce the solution formula (5) in Exercise 3. (I recommend setting the initial temperature equal to "*u0*", rather than using a subscript "u_0", since subscripted variables have caused problems in the past.) Don't worry if your solution isn't formatted exactly as it's displayed in equation (5), as long as you explain why the answers are equivalent.

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Notice that the indoor temperatures have a smaller amplitude, and that they lag the outdoor temperatures. The same thing happens with seasonal temperatures on earth, where the temperatures in a given hemisphere lag behind the times of maximal and minimal heating from the sun.

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Exercise 5: Use the cosine addition angle formula or Maple to show that the steady periodic solution given by equation (6) above, can be written in phase−amplitude form as

$$u_{sp}(t) = 80 - 6.0707 \cos\left(\frac{\pi\,(t - 7.5082)}{12}\right) \qquad (7)$$

Thus we conclude that the coolest and hottest indoor temperatures are around 7:30 a.m. and 7:30 p.m. As well, the amplitude of the temperature daily indoor temperature oscillation is approximately 6.07, vs. the outdoor amplitude of 10 degrees. Thus the indoor temperature is oscillating with a lag of about 3.5 hours compared to the outdoor one, and with smaller oscillations than outdoors. This is what our initial "dfield" plot predicted

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Now we're in Salt Lake City, it's winter, we're worried about pipes freezing after our heater breaks! So, the setup is that our heater is turned off at midnight, $t = 0$. Perhaps we forgot to pay our natural gas bill. At this time, the indoor temperature is $u_0 = 74$ degrees. The outdoor ambient temperature is given as in equation (1), except that the 24−hour temperatures vary between 21 and 49 degrees, with the high and low at 3 p.m. and 3 a.m., respectively. We don't have a very well insulated house, its insulation constant is $k = 0.32$.

Exercise 6: From the information above deduce the constants in the formula for the ambient temperature $A(t)$, and use Maple in any way you wish to find the solution to the initial value problem for the indoor temperature $u(t)$.

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Exercise 7: Create a Maple display which shows three plots: the Ambient temperature, the indoor temperature, and the steady−periodic indoor temperature, for this Salt Lake City winter problem.

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Exercise 8: Find the phase delay (the difference, in hours), between the ambient temperature and the indoor steady periodic temperature. Do this first by estimating from the display in Exercise 7 (clicking the mouse on a point of the display will yield the point coordinates in a small window at the top of your Maple program). Then verify the phase delay by writing the steady−periodic solution in phase−amplitude form as in equation (7) above, as discussed in class notes. (Use Maple for the computations).

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Exercise 9: Now suppose the inside temperature is 76 degrees when the furnace is turned off. Using 3−d plots and implicit curve plots, figure out for the range of insulation constants between $0.2 < k < 0.5$, and $0 < t < 72$ hours when the indoor temperature is at or below 30 degrees. (This could be bad for your water pipes.) Justify the logic and math used to find these "bad" times, in a short paragraph. Illustrate your work with a plot displays.

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