Math 2250-4
Friday Jan 21

HW for Fri 1/28
2.1 1, (3) (6, 8, 9, 12, 22, 33)
2.2 5, (7, 9) 12, (15) (14), (23)
(Maple Assignment 1) – see page 5 of today's notes

§2.1 Improved population models

let $P(t)$ = population (e.g. # people, # fish, etc.)
$B(t)$ = birth rate (e.g. people/year)
$D(t)$ = death rate (e.g. people/year)

$\beta(t) = \frac{B(t)}{P(t)}$ (people/year)/people e.g. .1 person/year per person.

fertility rate

$\delta(t) = \frac{D(t)}{P(t)}$ morbidity rate

Then

$$\frac{dP}{dt} = B(t) - D(t)$$

$$\frac{\left(\frac{dP}{dt}\right)}{P} = \beta(t) - \delta(t)$$

model 1 : $\beta(t) \equiv \beta$ const, $\delta(t) \equiv \delta$ const $\Rightarrow$ $\frac{dP}{dt} = (\beta - \delta)P = kP$

either exponential growth or decay, depending on sign of k ... we know this DE!

model 2 : $\beta = \beta_0 - \beta_1 P$ (β_0, β_1 const)
$\delta = \delta_0 + \delta_1 P$

$\beta_1 > 0$ "sophisticated" population *
$\delta_1 > 0$ ~ finite resources, disease, fighting

So

$$\frac{dP}{dt} = (\beta - \delta)P$$
$$= [(\beta_0 - \beta_1 P) - (\delta_0 + \delta_1 P)]P$$
$$= [(\beta_0 - \delta_0) - (\beta_1 + \delta_1)P]P$$

$$\frac{dP}{dt} = kP(M - P)$$
$$\frac{dP}{dt} = aP - bP^2$$

$k = \beta_1 + \delta_1$, $M = \frac{\beta_0 - \delta_0}{\beta_1 + \delta_1}$

$a = kM$, $b = k$ $\left(\begin{matrix} k = b \\ M = a/b \end{matrix}\right)$

Logistic eqtn

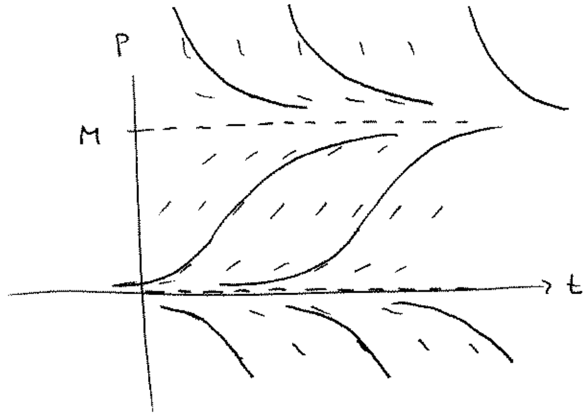
(There is a model 3 we'll discuss Monday for which $\beta = \beta_0 + \beta_1 P$ ("alligators") which can lead to the "doomsday-extinction" de, $\frac{dP}{dt} = -aP + bP^2$.)

What does the slope field predict for sol'ns to

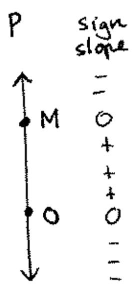
$$\text{IVP}\begin{cases}\dfrac{dP}{dt} = kP(M-P) & (k, M > 0)\\ P(0) = P_0\end{cases}$$

notice isoclines are horizontal lines in the (t,P) plane:

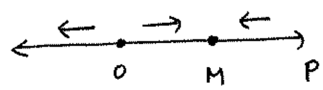
$P = 0 \Rightarrow m = 0$
$P = M \Rightarrow m = 0$



slope field picture



leads to phase portrait:



$P = 0$ and $P = M$ are called <u>equilibrium solutions</u> (because they are constant in time)

- It appears that any IVP sol'n $P(t)$ with $P(0) = P_0 > 0$ will converge to $P = M$.

  $M :=$ carrying capacity.

Solve the logistic IVP by separating variables (Use back of page if necessary!) (Then analyze solution to verify slope field & phase portrait predictions)

Sol: $P(t) = \dfrac{M}{\left(\frac{M-P_0}{P_0}\right)e^{-Mkt} + 1}$

Math 2280–1 and 2250–4
Friday January 21, 2011

Examples 4,5 from section 2.1 of the text, pages 83–84.

The Belgian demographer P.F. Verhulst introduced the logistic model around 1840, as a tool for studying human population growth. Our text demonstrates its superiority to the simple exponential growth model, and also illustrates why mathematical modelers must always exercise care, by comparing the two models to actual U.S. population data. here are actual U.S. populations by decade, from 1800 –2000, see e.g. the table on page 84:

```
> restart:  #clear Maple memory
> pops:= [[1800,5.3],[1810,7.2],[1820,9.6],[1830,12.9],
         [1840,17.1],[1850,23.2],[1860,31.4],[1870,38.6],
         [1880,50.2],[1890,63.0],[1900,76.2],[1910,92.2],
         [1920,106.0],[1930,123.2],[1940,132.2],[1950,151.3],
         [1960,179.3],[1970,203.3],[1980,225.6],[1990,248.7],
         [2000,281.4]]:
```

Unlike Verhulst, the book uses data from 1800, 1850 and 1900 to get constants in our two models. We let t=0 correspond to 1800.

Exponential Model: For the exponential growth model $P(t) = P_0 e^{rt}$ we use the 1800 and 1900 data to get values for P_0 and r:

```
> P0:=5.308;
  solve(P0*exp(r*100)=76.212,r);
```

$$P0 := 5.308$$

$$0.02664303814 \quad (1)$$

```
> P1:=t->5.308*exp(.02664*t);#exponential model -eqtn (9) page 83
```

$$P1 := t \to 5.308\, e^{0.02664\, t} \quad (2)$$

Logistic Model: We get P_0 from 1800, and use the 1850 and 1900 data to find k and M:

```
> P2:=t->M*P0/(P0+(M-P0)*exp(-M*k*t));
    #logisitic function, with our P0, eqtn (7) page 82
```

$$P2 := t \to \frac{M\, P0}{P0 + (M - P0)\, e^{-M k t}} \quad (3)$$

```
> solve({P2(50)=23.192, P2(100)=76.212},{M,k});
```

$$\{M = 188.1208275,\ k = 0.0001677157274\} \quad (4)$$

```
> M:=188.1208275;
  k:=.1677157274e-3;
  P2(t);  #should be our logistic model function,
    #equation (11) page 84.
```

$$M := 188.1208275$$

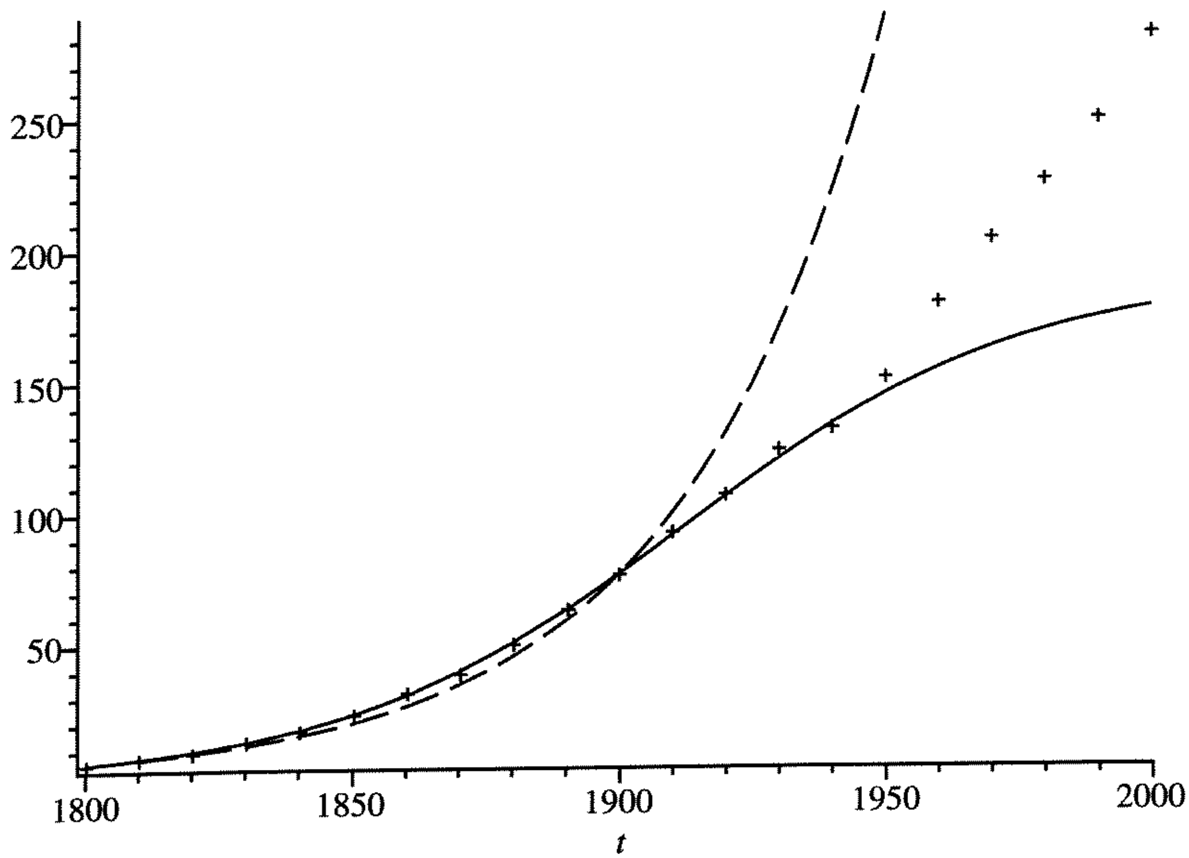
$$k := 0.0001677157274$$

$$\frac{998.5453524}{5.308 + 182.8128275\, e^{-0.03155082142\, t}} \quad (5)$$

Now compare the two models with the real data, and discuss:

```
> with(plots):
  plot1:=plot(P1(t-1800),t=1800..1950,color=black, linestyle=3):
    #this linestyle gives dashes for the exponential curve
  plot2:=plot(P2(t-1800),t=1800..2000,color=black):
  plot3:=pointplot(pops,symbol=cross):
  display({plot1,plot2,plot3},title=`U.S. population data
  and models`);
```

U.S. population data
and models



The exponential model takes no account of the fact that the U.S. has only finite resources. Any ideas on why the logistic model begins to fail (with our parameters) around 1950?

Math 2250/2280
First Maple Assignment
January 21, 2011

Find, use and modify commands to do the following:

0) Begin by downloading this document and opening it from Maple. A shortcut is to use the "open URL" option in the "File" menu item. The URL's below give links to this document and to a list of commands you might find helpful.

http://www.math.utah.edu/~korevaar/2250spring11/maplehw1.mw
http://www.math.utah.edu/~korevaar/2250spring11/maplecommands.mw

(The general way to find commands that will do what you want is to use Maple Help, look for the options on the left–hand menu, cut and paste from a related document, and/or use google.)

Now that you've opened this document from Maple, insert text and execute math commands as appropriate to create well–formated and clear responses to the following questions.

1) Use Maple to define the function $F(x) = \cos(3\,x+1) + e^{-x} + 1$.

2) Have maple compute the derivative of $F(x)$ and then define this derivative function to be $f(x)$.

3) Have maple antidifferentiate $f(x)$. Note that Maple does not add an additive constant when it antidifferentiates, so that your result won't exactly equal $F(x)$.

4) Use Maple to verify the Fundamental Theorems of Calculus for these examples, namely

$$\int_a^b f(x)\,dx = F(b) - F(a) \text{ and}$$

$$\frac{\partial}{\partial x}\left(\int_a^x f(r)\,dr\right) = f(x).$$

5) Create a single plot which contains the graphs of $f(x)$ and $F(x)$. Explain how the two graphs are related.

6) Define the 2–variable function $g(x, y) = \sin(y)\,e^{0.5\,x}$. Compute its x and y partial derivatives with Maple.

7) Create a single display containing the graph $z = g(x, y)$, and the horizontal plane $z = 1$. Use the square domain $-\pi < x < \pi$, $-\pi < y < \pi$. Include boxed axes, orient your display, and pick other options to make the picture as informative as possible.

8) Plot the part of the curve in the x–y plane which is given implicitly by the equation $1 = \sin(y)\,e^{0.5\,x}$, in the same x–y square as (7). Explain what this curve has to do with your 3–d display in (7).