

Name.....

I.D. number.....

Math 2250–4
FINAL EXAM
May 4, 2011

This exam is closed–book and closed–note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. Laplace Transform tables are included with this exam. **In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 150 points possible, and the point values for each problem are indicated in the right–hand margin. **Good Luck!**

problem	score	possible
1	_____	25
2	_____	15
3	_____	15
4	_____	25
5	_____	25
6	_____	20
7	_____	25
total	_____	150

1) Your father has a "sporty" car of mass 1200 kg and the engine and transmission of this car are able to provide a maximum force of $18,000 \text{ N}$, at all speeds.

1a) Assume the frictional forces on the car from air and road resistance are proportional to velocity, i.e. we are studying a linear drag model. What is the coefficient of friction, and what are its units, if Newton's law implies the car's velocity under the maximum acceleration force above satisfies the differential equation below?

$$v'(t) = 15 - 0.3 \cdot v(t)$$

(5 points)

1b) Suppose you're driving your father's car at a raceway, at a velocity of $30 \frac{m}{s}$, at which point you begin to accelerate at the maximum rate possible, i.e. the car's velocity satisfies the initial value problem

$$\begin{aligned} v'(t) &= 15 - 0.3 \cdot v(t) \\ v(0) &= 30. \end{aligned}$$

We've studied four techniques which apply to find solutions to this initial value problem: first order linear; separable; n^{th} -order linear (Chapter 5); Laplace transform. Use any two of these four techniques above to solve the initial value problem. If you try more than two techniques, indicate clearly which two you wish to have graded.

(20 points)

2a) Use Laplace transform techniques to find the solution to the initial value problem

$$x''(t) + \omega_0^2 \cdot x(t) = F_0 \sin(\omega_0 t)$$

$$x(0) = x_0$$

$$x'(0) = v_0$$

(12 points)

2b) What phenomenon is exhibited by solutions to (2a)? Explain.

(3 points)

3) Consider the matrix

$$A := \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix}$$

Find a basis of \mathbb{R}^2 made out of eigenvectors for this matrix.

(15 points)

4) We first encountered a tank cascade in Chapter 1, and this should be the last time we encounter one for at least several months. Consider two tanks. Salt water flows into the first tank at a constant rate of 800 gallons an hour, with a concentration of 1 pound of salt per 200 gallons of water. This first tank maintains a constant volume of 400 gallons by continuously pumping well-mixed water into the second tank, at the same constant rate of 800 gallons/hour. The second tank pumps well-mixed water out at this same rate, maintaining a constant volume of 800 gallons.

4a) Let $x(t), y(t)$ denote the amount of salt in tanks 1 and 2, respectively. Show that this tank system is modeled by the differential equations

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

(5 points)

4b) Use your eigenvalue/eigenvector work from problem (3) to exhibit the general solution to the homogeneous system of differential equations

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

(5 points)

4c) Find a particular solution to the non-homogeneous system of differential equations in (4a). (Hint: find a constant solution.)

(5 points)

4d) Suppose there was initially no salt in either tank, so that $x(0) = y(0) = 0$. Solve the resulting initial value problem for (4a).

(10 points)

5) Consider the following configuration of springs, with positive displacements from equilibrium measured to the right, as indicated.

5a) Derive the system of second order differential equations which models this system. Assume there are no frictional or external forces.

(5 points)

5b) Assume that in appropriate units, $m_1 = 2$, $m_2 = 2$, $k_1 = 4$, $k_2 = 6$. Show that in this case the system above reduces to

$$\begin{bmatrix} x''(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

(5 points)

5c) Find the general solution to the system in (5b). Describe the two fundamental modes of vibration.

(15 points)

6) Although we usually use a mass–spring configuration to give context for studying second order differential equations, we’ve also used the rigid–rod pendulum to effectively exhibit several key ideas from this course. Recall that in the undamped version of this configuration, we let the pendulum rod length be L , assume the rod is massless, and that there is a mass m attached at the end on which the vertical gravitational force acts with force $m \cdot g$. This mass will swing along circular arcs of signed arclength $s = L \cdot \theta$ from the vertical, where θ is the angle in radians from vertical. The configuration is indicated below. Note that we are considering a pendulum for which the mass (and rod) are able to rotate freely about the fixed end of the rod.

6a) Use the fact that the undamped pendulum system is conservative, i.e. the sum of the kinetic energy from the mass motion and the potential energy from its change in height must be constant once the system is put into motion, to derive the differential equation for $\theta(t)$,

$$\theta''(t) + \frac{g}{L} \cdot \sin(\theta(t)) = 0.$$

Hint: Since the total energy is constant, its time derivative must be zero.

(10 points)

6b) Under the assumption that $|\theta(t)|$ is near zero we linearized the $\sin(\theta)$ term in (6a) to get a linear differential equation. What is this differential equation?

(5 points)

6c) Use the linearized differential equation from (6b) to derive the formula we used for the period of a freely oscillating pendulum with small amplitude, in terms of g and L .

(5 points)

7) Assuming damping proportional to velocity, the pendulum equation in (6) is replaced with

$$\theta''(t) + c \cdot \theta'(t) + \frac{g}{L} \cdot \sin(\theta(t)) = 0$$

with positive constant c .

7a) Explain precisely how this second order damped pendulum differential equation above is related to the first order system of differential equations

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} y \\ -\frac{g \sin(x)}{L} - c \cdot y \end{bmatrix}$$

(4 points)

7b) Find all equilibrium solutions of the system (7a), and describe the pendulum configurations they correspond to.

(4 points)

7c) Consider a special case of the system above, in which the various constants yield the system

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} y \\ -5 \cdot \sin(x) - 2 \cdot y \end{bmatrix}.$$

Find the linearized system of differential equations near the the equilibrium solution $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(5 points)

7d) Compute eigenvalues of the linearized system in order to classify the stability and type of equilibrium at the origin. Would you consider this an "underdamped" or "overdamped" pendulum? (6 points)

7e) Use your work in (7d) to sketch a qualitatively correct picture of what the phase portrait for the system in (7c) looks like near the origin. (6 points)

